Sheaves of modules

Monday, November 4, 2024 5:10 PM

start of 06.11.24 lec

$(X, \mathcal{O}_X)$ be a ringed space.

Notation. $\mathrm{Mod}_{\mathcal{O}_X}$ = category of sheaves of $\mathcal{O}_X$-mods on X. =: category of $\mathcal{O}_X$-mods.

objects of $\mathrm{Mod}_{\mathcal{O}_X}$ = sheaves of $\mathcal{O}_X$-mods on X

A morphism $f: \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-mods is a map of sheaves s.t. $f_{|U}: \mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$ lin for $U \underset{\text{open}}{\subseteq} X$. Such a morphism is called $\mathcal{O}_X$-linear.

e.g.: $g: Y \to X$ morphism of schemes

$g_* \mathcal{O}_Y \in \mathrm{Mod}_{\mathcal{O}_X}$, $g^\#: \mathcal{O}_X \to g_* \mathcal{O}_Y$ is $\mathcal{O}_X$-lin by definition.

§ Operation on $\mathrm{Mod}_{\mathcal{O}_X}$.

- Given $\mathcal{O}_X$-linear $f: \mathcal{F} \to \mathcal{G}$, $\ker f$, $\operatorname{coker} f$, $\operatorname{im}(f)$ are in $\mathrm{Mod}_{\mathcal{O}_X}$.

- $\mathrm{Mod}_{\mathcal{O}_X}$ admits $\otimes_{\mathcal{O}_X}$ pdt.

  $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) :=$ sheafification of the presheaf $U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$

- Admits an internal hom sheaf

  $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$

- $\mathrm{Mod}_{\mathcal{O}_X}$ admits arbitrary direct sum and product

$$\boxed{\text{Ex: Check } \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})) \text{ in } \mathrm{Mod}_{\mathcal{O}_X}.}$$

- i) $(f, f^\#): (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a map of ringed spaces. For $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_Y}$, $f_* \mathcal{F}$ is naturally an $\mathcal{O}_X$-mod: $f_* \mathcal{F}$ is a $f_* \mathcal{O}_Y$ module; Then $f_* \mathcal{O}_Y$ is an $\mathcal{O}_X$-mod via the map $f^\#: \mathcal{O}_X \to f_* \mathcal{O}_Y$.

- ii) $(f, f^\#): (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ map of ringed spaces.

$f^\#: \mathcal{O}_X \to f_*\mathcal{O}_Y$.

Def. $(f, f^\#): (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ map of ringed spaces,
$\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$. The $f^*\mathcal{G}$ = pull back of $\mathcal{G}$ via f
is $f^{-1}\mathcal{G} \otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_Y$

[$\mathcal{O}_X \to f_*\mathcal{O}_Y$, gives $f^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y$, The $\otimes$ is taken on the ringed space $(Y, f^{-1}\mathcal{O}_X)$]

Rmk. 1) $f^*\mathcal{G}$ ~~is~~ $\in \mathrm{Mod}_{\mathcal{O}_Y}$ (Ex).

2) $(Y, \mathcal{O}_Y)$, $(X, \mathcal{O}_X)$ locally ringed spaces. (eg schemes).

$$(f^*\mathcal{G})_y = \mathcal{G}_{f(y)} \otimes_{\mathcal{O}_{X,f(y)}} \mathcal{O}_{Y,y}$$

Recall. $f^\#: \mathcal{O}_X \to f_*\mathcal{O}_Y$ ~~gi~~ induces $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$.

§ $\tilde{}$ construction.

Let A be a ring, $M \in \mathrm{Mod}_A$. Consider the presheaf $\tilde{M}$ on $\mathrm{Spec}(A)$.

$$\tilde{M}(U) = \Big\{ \text{set maps } s: U \to \coprod_{p \in U} M_p \;\Big|\; \begin{array}{l} \text{(i) } s(p) \in M_p \\ \text{(ii) for } p \in U, \exists\, p \in V \subseteq_{\text{open}} U \\ \text{and } m \in M \text{ s.t.} \\ s(q) = m/f \in M_q \\ \text{for some } f \notin \bigcup_{q \in V} q \end{array} \Big\}$$

Thm (Prop 5.1, Hart)

a) $\tilde{M}$ is an $\mathcal{O}_X$-mod.

b) For $p \in \mathrm{Spec}(A)$, $(\tilde{M})_p \xrightarrow{\sim} M_p$ as A_p modules

c) For $f \in A$, The natural map $M_f \xrightarrow{\sim} \tilde{M}(D(f))$ is an isom.

d) In particular $\Gamma(\mathrm{Spec}(A), \tilde{M}) \leftarrow M$ is an isom.

Pf. (a), (b) clear, note (c) $\Rightarrow$ (d)

(c) Lemma: Let $s \in \Gamma(\tilde{M}, \mathrm{Spec}(A))$ such that $s|_{D(g)} = 0$ for some $g \in A$. Then $g^m \cdot s = 0$ for some $m \in \mathbb{N}_{\geq 0}$.

g_i such that $s = m_i/g_i$ on $D(g_i)$

$g^m \cdot s = 0$ for some $m \in \mathbb{N}_{\geq 0}$.

Pf. Choose $g_1, g_2, \dots g_r$ such that $s = m_i/g_i$ on $D(g_i)$ and $\bigcup_{i=1}^{r} D(g_i) = \operatorname{Spec}(A)$.

Since $s|_{D(g_i) \cap D(g)} = s|_{D(g_i g)} = 0$

$m_i/g_i = 0 \in M_q \quad q \in D(g_i g)$

$\Rightarrow m_i/g_i = 0 \in M_{g_i g} = M_{g_i}[1/g]$

$\Rightarrow g^{t_i} \cdot \frac{m_i}{g_i} = 0$ in A_{g_i} for some t_i

$\Rightarrow g^{t_i} \cdot s|_{D(g_i)} = 0$

Take $m = \max\{t_1, t_2 \dots t_r\}$.

Then $g^m \cdot s|_{D(g_i)} = 0 \ \forall i \Rightarrow g^m \cdot s = 0$ ◻

Thm, c Given $s \in \tilde{M}(D(f))$, choose $g_1, g_2, \dots, g_r$ such that
- $D(f) = \bigcup_{i=1}^{r} D(g_i)$ and
- On $D(g_i)$, $s = m_i/g_i$ for some $m_i \in M$.

i.e. $g_i s = \frac{m_i}{1}$ on $D(g_i)$

By Lemma $g_i^{n_i+1} s. = g_i^{n_i} \cdot m_i \in \Gamma(\operatorname{Spec} A[1/f], \tilde{M})$

for some n_i $(*)$

$\bigcup_{i=1}^{r} D(g_i) = \bigcup_{i=1}^{r} D(g_i^{n_i+1}) = D(f)$

$\Rightarrow V(f) = V(g_1^{n_1+1}, g_2^{n_2+1} \dots, g_r^{n_r+1})$

$\Rightarrow f \in \sqrt{(g_1^{n_1+1} \dots g_r^{n_r+1})}$

$\Rightarrow f^m = g_1^{n_1+1}\beta_1 + \dots + g_r^{n_r+1}\beta_r$ for some m & $\beta_1 \dots \beta_r \in A$.

in $\Gamma(D(f), \tilde{M})$

$(*) \Rightarrow f^m s = \left(\sum g_i^{n_i+1}\beta_i\right) s \overset{\downarrow}{=} \sum g_i^{n_i}\beta_i \cdot m_i \in M$

$\Rightarrow s \in \operatorname{Im}(M_f \longrightarrow \tilde{M}(D(f)))$.

Rmk. Given $M \in \mathrm{Mod}_A$, consider the presheaf $\mathcal{M}$ of $\mathcal{O}_X$ modules where $X = \operatorname{Spec} A$:

$\mathcal{M}(V) = \{ m/f \mid f \notin \cup q, q \in V \}$.

of $\mathcal{O}_X$ module.

$$\mathcal{M}(V) = \{ m/f \mid f \notin \mathfrak{U}_q,\ q \in V \}.$$

Then $\tilde{M}$ is the sheafification of $\mathcal{M}$.

Thm. Given an A-mod map $f : M \to N$, one naturally gets an $\mathcal{O}_{\operatorname{Spec} A}$ linear map $\tilde{f} : \tilde{M} \to \tilde{N}$. Moreover $\tilde{id_M} = id_{\tilde{M}}$ and $\widetilde{f \cdot g} = \tilde{f} \cdot \tilde{g}$.

So $M \mapsto \tilde{M}$ is a functor from the category of A-modules to the category of $\mathcal{O}_{\operatorname{Spec}(A)}$-modules.

Thm. The functor $M \mapsto \tilde{M}$ above has the following properties.

(i) The natural map $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec}(A)}}(\tilde{M}, \tilde{N})$ is an isom.

(ii) $X = \operatorname{Spec} A$, $\mathcal{F} \in \operatorname{Mod}_{\mathcal{O}_X}$, $\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}) \to \operatorname{Hom}_A(M, \Gamma(X, \mathcal{F}))$ is an isom. Conversely for $\mathcal{G} \in \operatorname{Mod}_{\mathcal{O}_X}$, if the natural map $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}) \xrightarrow{\Psi^{\mathcal{G}}_{\mathcal{F}}} \operatorname{Hom}_A(\Gamma(X, \mathcal{G}), \Gamma(X, \mathcal{F}))$ is an isomorphism, then $\mathcal{G} \xrightarrow{\sim} \tilde{M}$ for some $M \in \operatorname{Mod}_A$.

(iii) $\widetilde{\left(\bigoplus_{i \in I} M_i \right)} = \bigoplus_{i \in I} \tilde{M}_i$ in $\operatorname{Mod}_{\mathcal{O}_{\operatorname{Spec}(A)}}$.

(iv) $\widetilde{M \otimes_A N} \xrightarrow{\sim} \tilde{M} \otimes_{\mathcal{O}_{\operatorname{Spec}(A)}} \tilde{N}$.

(v) $\varphi : A \to B$ ring homo, $M \in \operatorname{Mod}_A$, $(\varphi^{\#})^{*}(\tilde{M}) \xrightarrow{\sim} \widetilde{M \otimes_A B}$ in $\operatorname{Mod}_{\operatorname{Spec} B}$

(vi) φ as in (v), $N \in \operatorname{Mod}_B$, $(\varphi^{\#})_* \tilde{N} = \tilde{N}$ where on the right N is considered as an A-mod via restriction of scalars.

Pf (ii) sufficient cond^n ensuring $\mathcal{G} \xrightarrow{\sim} \tilde{M}$.

Take $M = \Gamma(X, \mathcal{G})$. Our assumption gives a map $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \tilde{M}) \xrightarrow[\sim]{\Psi^{\mathcal{G}}_{\tilde{M}}} \operatorname{Hom}_A(\Gamma(X, \mathcal{G}), M)$ that $\Psi^{\mathcal{G}}_{\tilde{M}}(\varphi_1) = id_M$.

$\varphi_1 : \mathcal{G} \to \tilde{M}$ such that $\Psi^{\tilde{M}}_{\mathcal{G}}$

The isom $\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{G}) \xrightarrow[\sim]{\Psi^{\tilde{M}}_{\mathcal{G}}} \operatorname{Hom}_A(M, M)$ gives

$\varphi_1 : \mathcal{G} \to M$ such ...

The isom $\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{G}) \xrightarrow[\sim]{\Psi^{\tilde{M}}_{\mathcal{G}}} \mathrm{Hom}_A(M, M)$ gives

$\varphi_2 : \tilde{M} \to \mathcal{G}$ such that $\Psi^{\tilde{M}}_{\mathcal{G}}(\varphi_2) = id_M$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{G}) \arrow[r, "\Psi^{\tilde{M}}_{\mathcal{G}}"] \arrow[d, "\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \varphi_1)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_A(M, M) \arrow[d, "\mathrm{Hom}(M, id) = id"] \\
\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{M}) \arrow[r, "\Psi^{\tilde{M}}_{\tilde{M}}"'] & \mathrm{Hom}(M, M)
\end{tikzcd}$$

So $\Psi^{\tilde{M}}_{\mathcal{G}}(\varphi_2) = \Psi^{\tilde{M}}_{\tilde{M}}(\varphi_1 \cdot \varphi_2) = id$

Since $\Psi^{\tilde{M}}_{\tilde{M}}$ is an isom $\varphi_1 \cdot \varphi_2 = id$.

Thm. $M, N \in \mathrm{Mod}_A$. $f : M \to N$ A-linear.

Then $\ker \tilde{f} \simeq \widetilde{\ker f}$, $\widetilde{\mathrm{coker} f} \xleftarrow{\sim} \mathrm{coker} \tilde{f}$

Pf. Have $\ker f \to M \to N$. This gives a complex $\widetilde{\ker f} \to \tilde{M} \to \tilde{N}$. This gives a map $\widetilde{\ker f} \to \ker \tilde{f}$, which is an isom at every stalk.

Thm. $0 \to M' \to M \to M'' \to 0$ is exact in Mod_A $\iff$ $0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ is exact in $\mathrm{Mod}_{\mathcal{O}_X}$.

§ Quasi coherent (QCoh) sheaves (ref stacks project tag 01BD)

$(X, \mathcal{O}_X)$ ringed space, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$

Def. $\mathcal{F}$ is QCoh if for every $x \in X$, $\exists$ a nbhd U of x and an exact seq

$$\bigoplus_{i \in I} \mathcal{O}_U \to \bigoplus_{j \in J} \mathcal{O}_U \to \mathcal{F}|_U \to 0$$

Thm. $X = \mathrm{Spec}(A)$, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$. $\mathcal{G}$ is QCoh iff $\mathcal{G} \xrightarrow{\sim} \tilde{M}$ in $\mathrm{Mod}_{\mathcal{O}_X}$.

Pf. $\Leftarrow$ Choose a presentation $\bigoplus_{j \in J} A \to \bigoplus_{i \in I} A \to M \to 0$

This gives an exact seq $\bigoplus_{j \in J} \mathcal{O}_X \to \bigoplus_{i \in I} \mathcal{O}_X \to \tilde{M} \to 0$.

$\Rightarrow$ Choose $f_1, f_2, \ldots f_n \in A$ such that $X = \bigcup_{j=1}^{n} D(f_j)$ and for each j, have an exact seq

$$\bigoplus \mathcal{O} \ldots \to \bigoplus \mathcal{O} \ldots \to \mathcal{G}|_{D(f_j)} \to 0$$

exact seq

$$\bigoplus_J \mathcal{O}_{D(f_j)} \longrightarrow \bigoplus_I \mathcal{O}_{D(f_j)} \longrightarrow \mathcal{G}|_{D(f_j)} \longrightarrow 0$$

Since $\mathcal{O}_{D(f_j)} = \widetilde{A[1/f_j]}$, They are Qcoh.

So the coker $\mathcal{G}|_{D(f_j)} \xrightarrow{\sim} \widetilde{\Gamma(D(f_j), \mathcal{G})}$

Let $i_j : D(f_j) \hookrightarrow X$, $i_{j_1 j_2} : D(f_{j_1}) \cap D(f_{j_2}) \hookrightarrow X$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; \| \; D(f_{j_1} f_{j_2})$

be the natural open immersions

We have an exact seq

$$0 \to \mathcal{G} \longrightarrow \bigoplus_j (i_j)_*(\mathcal{G}|_{D(f_j)}) \longrightarrow \bigoplus (i_{j_1 j_2})_*(\mathcal{G}|_{D(f_{j_1} f_{j_2})})$$

$(i_j)_*(\mathcal{G}|_{D(f_j)}) \cong \widetilde{\Gamma(D(f_j), \mathcal{G})}$ on X, $\quad (i_{j_1 j_2})_*(\mathcal{G}|_{D(f_{j_1} f_{j_2})}) \cong \widetilde{\Gamma(D(f_{j_1} f_{j_2}), \mathcal{G})}$ on X

$$(s_j) \longmapsto ((s_{j_1} - s_{j_2}))$$

$\mathcal{G}$ being the kernel, is also Qcoh.

· End of 6.11.24 lec

Thm: Let X be a scheme, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$.

(i) $\mathcal{G}$ is Qcoh iff for every affine open $U \subseteq X$, $\mathcal{G}|_U \xrightarrow{\sim} \tilde{M}_U$ for some $M_U \in \mathrm{Mod}_{\mathcal{O}_X(U)}$.

(ii) $\mathcal{G}$ is Qcoh iff $\exists$ an affine open covering $X = \cup U_\lambda$ s.t. $\mathcal{G}|_{U_\lambda} \xrightarrow{\sim} \tilde{M}_\lambda$ for some $M_\lambda \in \mathrm{Mod}_{\mathcal{O}_X(U_\lambda)}$

(iii) When X is affine checking $\mathcal{G} \simeq \tilde{M}$ iff $\mathcal{G}|_{U_\lambda} \simeq \tilde{M}_\lambda$ for some affine covr $X = \cup U_\lambda$ and $M_\lambda \in \mathrm{Mod}\, \mathcal{O}_X(U_\lambda)$.

Thm. X be a scheme, The category of Qcoh sheaves $\mathrm{QCoh}(X)$ is abelian, admits arbitrary direct sum.

Pf. HW:

§ Coherent Sheaves: (abv. coh)

The notion of coh sheaves on a scheme X is defined under the additional assumption X is

- The notion of coh sheaf is defined under the additional assumption X is locally noetherian (be careful, Hart assumes additionally X is noeth, which we do not).

<u>Def.</u> Let X be a locally noetherian scheme. $\mathcal{G} \in \mathrm{Mod}\,\mathcal{O}_X$. $\mathcal{G}$ is called <u>coherent</u> (abb coh) if one of the following equivalent cond^n are satisfied (eq. is HW)

(i) $\mathcal{G}$ is Qcoh and for every affine open $U \subseteq X$, $\Gamma(U, \mathcal{G})$ is a finitely generated $\mathcal{O}_X(U)$ mod.

(ii) There is an affine open covering $X = \bigcup_\lambda U_\lambda$ s.t. $\mathcal{G}|_{U_\lambda}$ is Qcoh and $\Gamma(U_\lambda, \mathcal{G})$ is a f.g $\mathcal{O}_X(U_\lambda)$ mod for $\forall \lambda$.

(iii) For every affine open $U \subseteq X$ or on each element U_λ of an affine open covering $X = \cup U_\lambda$, $\mathcal{G}|_U$ or $\mathcal{G}|_{U_\lambda}$ is isom to $\tilde{M}$ for some f.g $\mathcal{O}_X(U)$ or $\mathcal{O}_X(U_\lambda)$-mod M.

<u>Thm.</u> $\mathrm{QCoh}(X)$ is closed under taking ker, coker, $\oplus$, $\otimes$, $\mathrm{QCoh}(X)$ is an abelian category.

<u>Notation</u> $\mathrm{Coh}(X)$ = set of all coh $\mathcal{O}_X$-mods on X.

<u>Thm.</u> X, Y be noeth schemes.

(i) $\varphi : \mathcal{F} \to \mathcal{G}$ $\mathcal{O}_X$-lin map between coh sheaves. $\ker \varphi$, $\operatorname{coker} \varphi$ coh.

(ii) $f : Y \to X$, $\mathcal{F} \in \mathrm{Coh}(X)$, $f^*\mathcal{F} \in \mathrm{Coh}(Y)$.

(iii) If f is finite $\mathcal{G} \in \mathrm{Coh}(Y)$, $f_*\mathcal{G} \in \mathrm{Coh}(X)$.

(iv) $\psi : \mathcal{F} \to \mathcal{G}$ $\mathcal{O}_X$-lin $\mathcal{G} \in \mathrm{QCoh}(X)$, $\mathcal{F} \in \mathrm{Coh}(X)$. $\mathrm{Im}\,\psi \in \mathrm{Coh}(X)$.

(iv) ...

$\operatorname{Im}\varphi \in \operatorname{Coh}(X)$.

(v) $\mathcal{F}, \mathcal{G} \in \operatorname{Coh}(X)$, so is $\mathcal{F}\otimes\mathcal{G}$, $\mathcal{F}\oplus\mathcal{G}$,

<u>Thm.</u> $f: X \to Y$ map of schemes, f quasi-compact and separated

(i.e inverse image of any quasi-compact subset is quasi compact or for some affine open cover $Y = \cup U_i$, $f^{-1}(U_i)$ is quasi-compact }; when X is noetherian, any f is automatically q. compact).

$\mathcal{F} \in Q\operatorname{Coh}(X)$, $f_*\mathcal{F} \in Q\operatorname{Coh}(Y)$.

<u>Pf.</u> W.L.O.G Y is affine. Choose a finite affine covering $f^{-1}(Y) = \bigcup_{j=1}^{r} U_j$

As X is separated, $U_{j_1}\cap U_{j_2}$ is affine

We have an exact sq

$$0 \to \mathcal{F} \to \bigoplus (i_j)_*(\mathcal{F}|_{U_j}) \longrightarrow \bigoplus_{j_1,j_2} (i_{j_1,j_2})_*(\mathcal{F}|_{U_{j_1}\cap U_{j_2}})$$

$$(s_j) \longmapsto (s_{j_1} - s_{j_2})$$

Applying f_* <u>get an exact sq</u>

$$0 \to f_*\mathcal{F} \to \bigoplus f_*(i_j)_*(\mathcal{F}|_{U_j}) \longrightarrow \bigoplus f_*(i_{j_1,j_2})_*\mathcal{F}|_{U_{j_1}\cap U_{j_2}}$$

$f_*(i_j)_* = (f\circ i_j)_*$ $f\circ i_j : U_j \to Y$, both U_j, Y are affine, so the 2nd and 3rd terms are $Q\operatorname{Coh}$. So the kernel is $Q\operatorname{Coh}$.

Sheaves on Proj

Thursday, November 7, 2024 7:49 PM

Let $R = \bigoplus_{j \in \mathbb{N}} R_j$ be an $\mathbb{N}$-graded ring. We constructed a scheme $(X, \mathcal{O}_X) = (\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)})$; Set $R_+ = \bigoplus_{j>0} R_j$

Recall $\mathrm{Proj}(R) = \{ p \mid p \text{ homogeneous prime}, \ p \not\supseteq R_+ \}$

The top on $\mathrm{Proj}(R)$ has open basis $\{D_+(f)\}$ f homo of +ve deg

$D_+(f) = \{ p \in \mathrm{Proj}(R) \mid f \notin p \}$

$\mathcal{O}_X(D_+(f)) \cong (R[1/f])_0$.

§ Recall, a $\mathbb{Z}$-graded R mod is $M \in \mathrm{Mod}_R$ such that $M = \bigoplus_{\lambda \in \mathbb{Z}} M_\lambda$ as an ab group

and $R_{\lambda_1} \cdot M_{\lambda_2} \subseteq M_{\lambda_1 + \lambda_2}$, $\forall \lambda_1 \in \mathbb{N}, \lambda_2 \in \mathbb{Z}$.

- A graded R-lin map (or simply a graded map) between two graded mods is an R lin map $\varphi: M \to N$ s.t $\varphi(M_\lambda) \subseteq N_\lambda$ $\forall \lambda \in \mathbb{Z}$.
- The cat of $\mathbb{Z}$-graded R-mods with graded R-lin maps forms a cat denoted Mod_R^{gr}

Given $M \in \mathrm{Mod}_R^{gr}$, we construct a sheaf of $\mathcal{O}_X$-mods, denoted $\tilde{M}$ on $X = \mathrm{Proj}\ R$

Caution: This $\tilde{M}$ is not the q coh sheaf $\tilde{M}$ on $\mathrm{Spec}(A)$.

- Note given a mult closed set S of homogeneous elts of R, $S^{-1}R$ is a $\mathbb{Z}$-graded ring. $S^{-1}M$ is a $\mathbb{Z}$-graded mod / $S^{-1}R$.

$$(S^{-1}R)_\lambda = \{ r/s \in S^{-1}R \mid r, s \text{ homo}, \deg r - \deg s = \lambda \}$$

$$(S^{-1}M)_\lambda = \{ m/s \in S^{-1}M \mid m, s \text{ homo}, \deg m - \deg s = \lambda \}$$

- For $p \in \mathrm{Proj}(R)$, set S_p = homo elts of $R \setminus p$

$$M_{(p)} = (S_p^{-1} M)_0$$

Def of $\tilde{M}$: Consider the fun sheaf of $\mathcal{O}_X$-mods on X;

$$\tilde{M}(U) = \{ \text{set maps } s: U \to \coprod_{p \in U} M_{(p)} \mid \begin{array}{l} \cdot\ s(p) \in M_{(p)} \\ \cdot \text{ For every } U, \exists \text{ a open nbhd } V \subseteq U \text{ of } p, \text{ a homo } f \notin \bigcup_{q \in V} q \\ \ \ \& \ m \in M \text{ s.t } s(q) = m/f \in M_{(q)} \end{array} \}$$

It's implicit that $\deg m = \deg f$

Recall: $\tilde{R} = \mathcal{O}_X$, So for $s \in \mathcal{O}_X(U)$, $t \in \tilde{M}(U)$

$$(s\,t)(q) = \underset{\in R_{(q)}}{s(q)} \cdot \underset{\in M_{(q)}}{t(q)} \in M_{(q)} \quad \forall q \in U$$

Thm: 1) $\tilde{M}$ is a sheaf for any $M \in \mathrm{Mod}_R^{gr}$.
For $p \in \mathrm{Proj}\,R$, the natural map $M_{(p)} \to (\tilde{M})_p$ is an isom with inverse given by $s \mapsto s(p)$ for a section $s \in \tilde{M}(U)$, $p \in U$.

2) Given a graded R-lin map $\varphi: M \to N$, naturally have $\tilde{\varphi}: \tilde{M} \to \tilde{N}$

for $t \in \tilde{M}(U)$, $p \in U$: $\tilde{\varphi}_U(t)(p) = \varphi_{(p)}(t(p))$; $\varphi_{(p)}: M_{(p)} \to N_{(p)}$

3) $\widetilde{\varphi_1 \cdot \varphi_2} = \tilde{\varphi}_1 \cdot \tilde{\varphi}_2$, $\widetilde{id_M} = id_{\tilde{M}}$

4) So $M \to \tilde{M}$ is gives a functor $\mathrm{Mod}_R^{gr} \to \mathrm{Mod}_{\mathcal{O}_X}$.

Thm: For $M \in \mathrm{Mod}_R^{gr}$; $\tilde{M} \in \mathrm{QCoh}(X)$. Moreover $\Gamma(D_+(f), \tilde{M}) \cong (M[1/f])_0$ for

Thm: For $M \in \mathrm{Mod}_R^{gr}$, $\tilde{M} \in \mathrm{Qcoh}(X)$. Moreover $\Gamma(D_+(f), \tilde{M}) \cong (M[1/f])_0$ for every homog elt f of +ve deg.

Pf. Since being qcoh is a local property, it's enough to check that for any homog elt f of +ve deg $\tilde{M}|_{D_+(f)}$ is qcoh.

Set $\Gamma = M[1/f]_0$. The identity map $\Gamma \to \Gamma(D_+(f), \tilde{M})$ gives an $\mathcal{O}_{D_+(f)}$-lin map

$* \quad \tilde{\Gamma} \longrightarrow \tilde{M}|_{D_+(f)}$ as $\tilde{\Gamma}$ is qcoh on the affine $D_+(f)$.

Claim. $*$ is an isom.

Pf. We check isom at the stalks at $p \in D_+(f)$.

Recall that $D_+(f) \xrightarrow{\sim} \mathrm{Spec}(R[1/f]_0)$

$$q \longmapsto qR[1/f] \cap R[1/f]_0 = q_a$$

The map induced by $*$: $(M[1/f]_0)_{p_a} = \tilde{\Gamma}_{p_a} \longrightarrow \tilde{M}_p \xrightarrow{\sim} M_{(p)}$ $\quad **$

$$g \in S_p, \quad \frac{m/f^{a}}{g/f^{e_2}} \longmapsto \frac{m f^{e_2}}{g f^{a}}$$

Note $(M[1/f]_0)_{p_a} = \left[(S_p)^{-1} M[1/f]\right]_0 = (S_p^{-1}M)_0$

So $**$ is an isom.

Rmk. So $\tilde{M}$ can be thought to be obtained by gluing $\widetilde{M[1/f]_0}$ on $D_+(f)$ for different f's.

Thm. 1) $f: M \to N$ be a graded R-mod map.
$\ker \tilde{f} \xleftarrow{\sim} \widetilde{\ker f}$, $\mathrm{coker}\, \tilde{f} \xrightarrow{\sim} \widetilde{\mathrm{coker} f}$

2) If $0 \to M' \to M \to M'' \to 0$ is exact in Mod_R^{gr} [This includes the hypothesis that all the maps are graded]
Then $0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ is also exact in $\mathrm{Mod}\,\mathcal{O}_X$

3) $(\bigoplus_{i \in I} M_i)^{\sim} \xleftarrow[\text{in } \mathrm{Mod}\,\mathcal{O}_X]{\sim} \bigoplus_{i \in I} \tilde{M}_i$ for any set I.

Thm. 1) $\tilde{M} = 0 \iff \forall f$ homog of +ve deg $M[1/f]_0 = 0$
$\iff$ for a collection of homog elt of +ve deg $f_1, f_2, \ldots, f_r$ s.t. $X = \bigcup_{i=1}^{r} D_+(f_i)$ and any $m \in M_n$ where $\deg f_i \mid n \ \forall i$, $\exists\, n_1, n_2, \ldots, n_r$ s.t. $f_i^{n_i} \cdot m = 0$

[A special case is when one can choose f_i's of deg 1 s.t. $X = \bigcup_{i=1}^{r} D(f_i)$]

2) For $M \in \mathrm{Mod}_R^{gr}$, $\tilde{i}$ is an isom.
$i: \bigoplus_{\lambda \in \mathbb{N}} M_\lambda \hookrightarrow M$.

Pf. $\tilde{M} = 0 \iff \forall f$ homog of +ve deg $\tilde{M}|_{D_+(f)} = 0$
$\underset{\tilde{M}\text{ qcoh}}{\iff}$ ——"—— $M[1/f]_0 = 0$
$\iff$ for a collection $\{f_i\}_{i \in I}$, f_i homog of +ve deg s.t. $X = \bigcup D_+(f_i)$, $M[1/f_i]_0 = 0$

Now assume $X = \bigcup_{i=1}^{r} D(f_i)$

s.t $X = D_+(f_i)$, $M[1/f_i]_0$

Now assume $X = \bigcup_{i=1}^{r} D(f_i)$

$M[1/f_i]_0 = \{ m/f_i^n \mid \deg m = n \deg f_i \}$

$M[1/f_i]_0 = 0 \iff$ for $m \in M_\lambda$ s.t $\deg f_i \mid \lambda$

$m / f_i^{\lambda/\deg f_i} = 0 \in M[1/f_i]_0 \subseteq M[1/f_i]$

$\iff f_i^{n_i} m = 0$ for some n_i.

2) Consider the exact seq in Mod_R^{gr}

$$0 \to \bigoplus_{\lambda \in \mathbb{N}} M_\lambda \to \overset{M}{\overset{\|}{\bigoplus_{\lambda \in \mathbb{Z}} M_\lambda}} \to Q \to 0$$

Every nonzero homo elt in Q lifts to a non-zero homo elt, m, of $-$ve deg in M

Now for every homo elt, f, of R of $+$ve deg

$\deg f^n . m \geq 0$ for $n >> 0 \Rightarrow f^n . \underset{\text{im in } Q}{\underset{\uparrow}{\bar{m}}} = 0$

$\Rightarrow \tilde{Q} = 0$

$\Rightarrow 0 \to \widetilde{\bigoplus_{\lambda \in \mathbb{N}} M_\lambda} \to \tilde{M} \to 0$ is exact.

End of 08.11.24 Lec

§. $\mathcal{O}_X(n)$, $\mathcal{F}(n)$, ... $X = (\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)})$

Def. • For $M \in \mathrm{Mod}_R^{gr}$, $M(n) \in \mathrm{Mod}_R^{gr}$ is the object whose underlying R-mod is M, but

$$M(n)_m = M_{n+m}$$

• $\mathcal{O}_X(n) := \widetilde{R(n)}$ • For $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$

$\mathcal{F}(n)$ is called the n-th Serre twist of $\mathcal{F}$.

Note There is a map $M_0 \to \Gamma(X, \tilde{M})$ and so

$M_n = (M(n))_0 \to \Gamma(X, \widetilde{M(n)})$

Def: Y scheme, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_Y}$ is called loc. free$_{r_\alpha}$ if $\exists$ an open covering $Y = \bigcup_{\alpha \in I} U_\alpha$ s.t $\mathcal{F}|_{U_\alpha} \xrightarrow{\sim} \bigoplus \mathcal{O}_{U_\alpha}$ $r_\alpha \in \mathbb{N}$ in $\mathrm{Mod}_{\mathcal{O}_{U_\alpha}}$

Prop • There are natural maps $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \to \mathcal{O}_X(n+m)$ and $\mathcal{F}(n) \otimes \mathcal{O}_X(m) \to \mathcal{F}(n+m)$.

• Suppose $\deg f = m > 0$, Then

(i) $\mathcal{O}_{D_+(f)} \to \mathcal{O}_X(m)|_{D_+(f)}$, $1 \mapsto f/1$ is an isom in $\mathrm{Mod}_{\mathcal{O}_{D_+(f)}}$

(ii) $\mathcal{F}(n)|_{D_+(f)} \otimes \mathcal{O}_X(m)|_{D_+(f)} \xrightarrow{\sim} \mathcal{O}_X(n+m)|_{D_+(f)}$

Pf. (i) Using qcoh, enough to prove

$$\begin{array}{ccc} \Gamma(D_+(f), \mathcal{O}_X) & \longrightarrow & \Gamma(D_+(f), \mathcal{O}_X(m)) \\ \| & & \uparrow\wr \\ R[1/f]_0 & \longrightarrow & [R(m)[1/f]]_0 = (R[1/f])_m \\ 1 & \longmapsto & f/1 \end{array}$$

is an isom

$$R[1/f]_0 \longrightarrow \Gamma(\ldots)$$
$$1 \longmapsto f/1$$
is an isom, inverse $s/f \longleftarrow s$

(ii) Since $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n)$, enough to prove
$$\mathcal{O}_X(n)|_{D_+(f)} \otimes \mathcal{O}_X(m)|_{D_+(f)} \longrightarrow \mathcal{O}_X(n+m)|_{D_+(f)}$$
is an isom

q-coh $\Rightarrow$ enough to check the isom $x \otimes y \longmapsto xy$
$$R[1/f]_n \otimes_{R[1/f]_0} R[1/f]_m \longrightarrow R[1/f]_{n+m}$$
$$\downarrow\wr \qquad\qquad\qquad ||$$
$$R[1/f]_n \otimes_{R[1/f]_0} f \cdot R[1/f]_0 \xrightarrow{\sim} f \cdot R[1/f]_n$$

§ Given $\mathcal{F} \in \mathrm{QCoh}(X)$, $X = \mathrm{Proj}(R)$ we want to construct a candidate mod M s.t. $\widetilde{M} \xrightarrow{\sim} \mathcal{F}$.

Caution. Such an M need not exist.

Def. Given $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, set $\Gamma_*\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n))$
$$\Gamma_*\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}(n)$$

Prop
- $\Gamma_*\mathcal{F}$ is naturally a $\Gamma_*\mathcal{O}_X$ mod
- There is a map of graded rings $R \to \Gamma_*\mathcal{O}_X$.
- Thus $\Gamma_*\mathcal{F}$ is naturally an R-mod.
- There is an $\mathcal{O}_X$-lin map $\widetilde{\Gamma_*\mathcal{F}} \longrightarrow \mathcal{F}$

Pf
- $\Gamma_*\mathcal{O}_X = \bigoplus_{n\in\mathbb{N}} \Gamma(X, \mathcal{O}_X(n))$, have
$$\Gamma_*(X, \mathcal{O}_X(n)) \times \Gamma_*(X, \mathcal{O}_X(m))$$
$$\downarrow$$
$$\Gamma(X, \mathcal{O}_X(n+m))$$

making $\Gamma_*\mathcal{O}_X$ a ring.

Have
$$\Gamma(X, \mathcal{F}(n)) \times \Gamma(X, \mathcal{O}_X(m)) \longrightarrow \Gamma(X, \mathcal{F}(n+m)).$$
$$|| \qquad\qquad || \qquad\qquad ||$$
$$\Gamma_*\mathcal{F}_n \qquad \Gamma_*\mathcal{O}_{X\,m} \qquad \Gamma_*\mathcal{F}_{n+m}$$

- $$R_n \longrightarrow \Gamma(X, \mathcal{O}_X(n))$$
$$r \longmapsto r/1$$

- We describe $\widetilde{\Gamma_*\mathcal{F}}(D_+(f)) \longrightarrow \mathcal{F}(D_+(f))$

q-coh $\Rightarrow$ Enough to produce $\Gamma_*\mathcal{F}[1/f]_0 \longrightarrow \Gamma(D_+(f), \mathcal{F})$

Note $\Gamma_*\mathcal{F}|_{D_+(f)} = \bigoplus_{\lambda\in\mathbb{Z}} \Gamma(D_+(f), \mathcal{F}(\lambda))$ is

not a $\Gamma_*\mathcal{O}_{D_+(f)} = \bigoplus_{\lambda\in\mathbb{Z}} \Gamma(D_+(f), \mathcal{O}_X(m))$ mod.

In the later ring the image of f via
$$R \longrightarrow \Gamma_*\mathcal{O}_X$$
$$\searrow \qquad \downarrow$$
graded $\qquad \Gamma_*\mathcal{O}_{D_+(f)}$

is invertible as $1/f \in \Gamma(D_+(f), \mathcal{O}(-\deg f))$

$\downarrow \qquad 1/f \in \Gamma(D_+(f), \mathcal{O}(-\deg f))$

$\Gamma_* \mathcal{O}_{D_+(f)}$

Thus have a graded $R[1/f] \longrightarrow \Gamma_* \mathcal{O}_{D_+(f)}$. This makes $\Gamma_* \mathcal{F}|_{D_+(f)}$ an $R[1/f]$ module and induces a graded R-lin map, $\Gamma_* \mathcal{F}[1/f] \longrightarrow \Gamma_* \mathcal{F}|_{D_+(f)}$

Restrict to the deg 0 piece to obtain the desired map.

<u>Thm</u> Assume R is a finitely generated R_0 alg; i.e $\exists$ $g_1, g_2 \ldots g_r$ +ve deg homo elts s.t

$$R_0[x_1, \ldots x_r] \longrightarrow R \quad , R_0\text{-lin is sur.}$$
$$x_i \longmapsto g_i$$

- For $\mathcal{F} \in Qcoh(X)$, the map $\widetilde{\Gamma_* \mathcal{F}} \longrightarrow \mathcal{F}$ is an isom.

<u>Pf:</u> Our hypothesis $\Rightarrow$ $R_+ \underset{\text{ideal}}{=} (g_1, g_2 \ldots g_r)$

$$\Rightarrow X = \bigcup_{i=1}^{r} D_+(g_i)$$

Replace g_i by $g_i^{\prod_{j\neq i} \deg g_j}$ to assume $\deg g_i$ are all equal $= d$. Note $X = \bigcup_{i=1}^{r} D_+(g_i)$.

Note $\mathcal{O}_X(nd)$ is invertible for $\forall n$.

Since $\mathcal{F}$ is qcoh it is enough to check that the induced map $(\Gamma_* \mathcal{F}[1/g_i])_0 \longrightarrow \Gamma(D_+(g_i), \mathcal{F})$ $(+)$ is an isom $\forall i$

<u>Needed lemma</u>

<u>Lemma:</u> Y quasi compact scheme, $\mathcal{G} \in Qcoh(Y)$, $\mathcal{L}$ invertible, $s \in \Gamma(Y, \mathcal{L})$. Let

$$D_s = \{ y \in Y \mid s_y \notin m_y \mathcal{L}_y \} \quad m_y \text{ is the max ideal of } \mathcal{O}_{Y,y}.$$

(i) D_s is open

(ii) Given $t_1 \in \Gamma(Y, \mathcal{G})$ if $t_1|_{D_s} = 0$, $\exists$ $n \in \mathbb{N}$ s.t $t_1 \otimes s^n = 0 \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$

(iii) Assume intersection of every two open affines have a finite covering by affine opens in Y. [This is immediate if Y is (quasi) separated]

(iii) Assume intersection of every ... covering by affine opens in Y. [This is immediate if Y is noetherian or Y is (quasi) separated]

Given $t \in \Gamma(D_s, \mathcal{G})$, $\exists\ n \in \mathbb{N}$, $\tilde{t} \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$

s.t. $\underset{\Gamma(D_s, \mathcal{G} \otimes \mathcal{L}^n)}{t \otimes s^n} = \tilde{t}|_{D_s}$

Pf. (i) $\boxed{\text{Quasicompactness is unnecessary}}$. Choose an affine open covering $Y = \bigcup_j U_j$ s.t

$\mathcal{O}_{U_j} \xrightarrow{\sim} \mathcal{L}|_{U_j}$ $\quad 1 \longrightarrow s_j$, $s|_{U_j} = f_j s_j$ $\quad f_j \in \Gamma(U_j, \mathcal{O}_{U_j})$)

$D_s \cap U_j = D(f_j) \underset{\text{open}}{\subseteq} U_j$. Thus $D_s \cap U_j$ is open in U_j. $\forall j$. Thus D_s is open in X. ◻

· Since Y is quasi-compact, we choose a finite subcover of U_j $\quad Y = \bigcup_{j=1}^{r} U_j$. Fix this for (ii), (iii)

(ii) $t_1|_{U_j \cap D_s} = 0$, $U_j \cap D_s = D_{U_j}(f_j)$ [This means the basic affine open given by f_j inside U_j]

Since U_j is affine and $\mathcal{G}|_{U_j}$ is qcoh $\quad f_j^{n_j} . t_1 = 0 \in \Gamma(U_j, \mathcal{G}|_{U_j})$ for some n_j

$\implies t_1 \otimes s^{n_j}|_{U_j} = 0 \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^{n_j})$

Take $n = \max\{n_1, \dots, n_r\}$. Then $t_1 \otimes s^n|_{U_j} = 0 \quad \forall j$

$\implies t_1 \otimes s^n = 0 \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$

(iii) Since $D_s \cap U_j = D(f_j)$ in U_j and

$\mathcal{G}(D_s \cap U_j) = \mathcal{G}(U_j)[1/f_j]$ $\quad$ [$\because \mathcal{G}$ is qcoh]

$\exists\ n_j$ s.t $f_j^{n_j} . t$ is the restriction of a section in $\Gamma(\mathcal{G}, U_j)$ to $D(f_j) = D_s \cap U_j$. This means

$\exists\ t_j \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^{n_j})$ s.t $t \otimes s^{n_j} = t_j|_{D_s \cap U_j}$

Let $n_0 = \max\{n_1, n_2, \dots, n_r\}$

Set $t'_j = t_j \otimes s^{n_0 - n_j} \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^{n_0})$

On $U_{j_1} \cap U_{j_2} \cap D_s$ $\quad t'_{j_1} = t'_{j_2} = t|_{U_{j_1} \cap U_{j_2} \cap D_s} \otimes s^{n_0}|_{U_{j_1} \cap U_{j_2} \cap D_s}$

By our hypothesis $U_{j_1} \cap U_{j_2} = \bigcup_{\alpha \in I} V_\alpha^{j_1 j_2}$, where $|I| < \infty$.

Using (ii) on each $V_\alpha^{j_1 j_2}$ and finiteness of the covering $U_{j_1} \cap U_{j_2} = \bigcup V_\alpha^{j_1 j_2}$

$t'_{j_1} \otimes s^{n_{j_1 j_2}} = t'_{j_2} \otimes s^{n_{j_1 j_2}}$ $\quad n_0 + n_{j_1 j_2}$

by ... on each $V_\alpha^{j_1 j_2}$.

Using (ii) and finiteness of the covering $U_{j_1} \cap U_{j_2} = \cup V_\alpha$ find $n_{j_1 j_2}$ such that $t'_{j_1} \otimes s^{n_{j_1 j_2}} = t'_{j_2} \otimes s^{n_{j_1 j_2}} \in \Gamma(U_{j_1} \cap U_{j_2}, \mathcal{G} \otimes \mathcal{L}^{n_0 + n_{j_1 j_2}})$

set $n = \max_{j_1, j_2 \in I} \{ n_0 + n_{j_1 j_2} \}$

Then $t'_j \otimes s^n \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^n) \ \forall j$

and $t'_{j_1} \otimes s^n |_{U_{j_1} \cap U_{j_2}} = t'_{j_2} \otimes s^n |_{U_{j_1} \cap U_{j_2}} \ \forall j_1, j_2$

$\Longrightarrow \tilde{t} \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$ s.t $\tilde{t}|_{U_j} = t_j \otimes s^n \ \forall j$

Clearly on $\tilde{t}|_{D_s} = t \otimes s^n$ on $\Gamma(D_s, \mathcal{G} \otimes \mathcal{L}^n)$

Back to (†)

<u>injectivity.</u> If s/g_i^n goes to zero

$s|_{D_+(g_i)} = 0 \Longrightarrow \exists n_i$ s.t $s \cdot g_i^{n_i} = 0 \in \Gamma(X, \mathcal{F}(n_i d))$

$\Longrightarrow s/g_i^n = 0$ in $(\Gamma_* \mathcal{F}[1/g_i])_0$

<u>Surjectivity.</u> Note $X = \cup D_+(g_i)$

$D_+(g_\alpha) \cap D_+(g_\beta) = D_+(g_\alpha g_\beta)$

So we can apply (iii) of Lemma above on X.

Given $t \in \mathcal{F}(D_+(g_i))$. $\exists n_i \in \mathbb{N}$ s.t

$t \otimes g_i^{n_i} = \tilde{t}|_{D_+(g_i)}$ for some $\tilde{t} \in \Gamma(X, \mathcal{F}(n_i d))$

Then $\tilde{t}/g_i^{n_i} \in (\Gamma_* \mathcal{F}[1/g_i])_0$ with its image in $\mathcal{F}|_{D_+(g_i)}$ being t.

Thm: Assume that $\exists$ homog elts of +ve deg $g_1, g_2 \dots g_r$ s.t $R = R_0[g_1, g_2 \dots g_r]$ and $\mathrm{Proj}(R) = X$ is locally noeth. Given $\mathcal{F} \in \mathrm{Coh}(X)$, $\exists$ a <u>finitely gen</u> $M \in \mathrm{Mod}_R^{gr}$ s.t $\tilde{M} \xrightarrow{\sim} \mathcal{F}$

<u>Pf.</u> Choose $n_1, n_2, \dots n_r$ s.t $\deg g_i^{n_i} = \deg g_j^{n_j} = d \ \forall i, j$

Then $X = \bigcup_{i=1}^{r} D_+(g_i^{n_i})$. So $\mathcal{O}_X(d)$ is invertible.

Realize $g_i^{n_i} \in \Gamma(X, \mathcal{O}_X(d))$. Note $D_{g_i^{n_i}} = D_+(g_i^{n_i})$

$g_i^{n_i}$ is thought of in $\Gamma(X, \mathcal{O}_X(d))$

Since $\mathcal{F}$ is coh, $\Gamma(D_+(g_i^{n_i}), \mathcal{F})$ is a f.g $\Gamma(D_+(g_i^{n_i}), \mathcal{O}_X)$

$\Gamma(X, \mathcal{O}_X(d))$

Since $\mathcal{F}$ is coh, $\Gamma(D_+(g_i^{n_i}), \mathcal{F})$ is a f.g $\Gamma(D_+(g_i^{n_i}), \mathcal{O}_X)$ mod. By the lemma above, $\exists\, s_1, s_2, \ldots, s_{r_i} \in \Gamma(X, \mathcal{F}(d_i d))$ such that $\{s_d / g_i^{n_i d_i}\}_{d=1,\ldots,r_i}$ is a set of gen.

By varying i and possibly increasing d_i, we can find $m \in \mathbb{N}$ & finitely many elts $t_1, t_2, \ldots, t_n \in \Gamma(X, \mathcal{F}(dm))$. s.t $\forall i$, $\{t_i / g_i^{n_i m}\}_{i=1,\ldots,n}$ generate the $\Gamma(D_+(g_i^{n_i m}), \mathcal{O}_X)$ mod $\mathcal{F}(D_+(g_i^{n_i}))$.

Let M be the (finitely generated) R-submodule of $\Gamma_* \mathcal{F}$ generated by $t_1, t_2, \ldots, t_n$.

Claim: The inclusion map $M \hookrightarrow \Gamma_* \mathcal{F}$ induces an isom $\tilde{M} \longrightarrow \widetilde{\Gamma_* \mathcal{F}}$ in $\mathrm{Mod}_{\mathcal{O}_X}$.

Pf It's enough to check that for each i the induced map

$$\begin{array}{ccc} \Gamma(D_+(g_i^{n_i}), \tilde{M}) & \longrightarrow & \Gamma(D_+(g_i^{n_i}), \widetilde{\Gamma_* \mathcal{F}}) \\ \uparrow\wr & & \uparrow\wr \\ (M[1/g_i^{n_i}])_0 & \longrightarrow & (\Gamma_* \mathcal{F}[1/g_i^{n_i}])_0 \end{array}$$

is an isom.

The injectivity follows as $M \subseteq \Gamma_* \mathcal{F}$; surjectivity follows from the diag

$$\begin{array}{ccc} (M[1/g_i^{n_i}])_0 & \longrightarrow & \mathcal{F}(D_+(g_i^{n_i m})) \\ & \searrow & \nearrow \wr \\ & & (\Gamma_* \mathcal{F}[1/g_i^{n_i m}])_0 \end{array}$$

where the top arrow is surj by construction.

Invertible sheaves:

Def: X be a scheme. A locally free sheaf of rk 1 is called an invertible sheaf.

Prop: i) Let $\mathcal{L}$ be an invertible sheaf. Then

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \longrightarrow \mathcal{O}_X \qquad (\varphi \otimes s) \mapsto \varphi(s)$$

is an isom

ii) For an $\mathcal{O}_X$-mod $\mathcal{F}$, suppose there is an $\mathcal{O}_X$-mod $\mathcal{G}$ and an isom $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{O}_X$. Then $\mathcal{F}$ is invertible.

Pf (i) Given $x \in X$, choose an open nbhd U of x s.t $\mathcal{L}|_U \xleftarrow{\sim} \mathcal{O}_U$. We have a diag using this isom

$$\begin{tikzcd}
 & \underline{Hom}_{O_X}(\mathcal{L}, O_X)|_U \arrow[d, "\wr"] & \otimes_{O_U} & \mathcal{L}|_U \arrow[r] & O_U \arrow[d, equal, "id"] \\
O_U \arrow[r, "\sim"] & \underline{Hom}_{O_U}(O_U, O_U) & \otimes & O_U \arrow[u, "\wr"] \arrow[r] & O_U
\end{tikzcd}$$

Since the bottom row is an isom, we are done.

(ii) Check at stalks.

Prop. X be a scheme. The isom classes of invertible O_X-mods form an abelian group under $\otimes$ operation, denoted $\mathrm{Pic}(X)$ — called the Picard group or the group of invertible sheaf.

Prop\Eg. • R be an $\mathbb{N}$-graded ring. Suppose $\exists\, g_1, g_2, \ldots g_r$ each of deg d such that $\mathrm{Proj}(R) = \bigcup_{i=1}^{r} D_+(g_i)$.
Then for each n, $O_X(nd)$ is invertible.

- So if $d=1$, each $O_X(n)$ invertible
- Assume R is gen over R_0 by deg 1 elt as an alg (i.e R standard graded), then $d=1$ and each $O_X(n)$ is invertible.

Eg: We will see $\mathrm{Pic}(\mathbb{A}^n_{\mathbb{C}}) \cong \{id\}$, $\mathrm{Pic}(\mathbb{P}^n_{\mathbb{C}}) \xrightarrow{\sim} \mathbb{Z} \cdot O(1)$
$\mathbb{P}^n_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[x_0 .. x_n])$

End of 13.11.24 Lecture.

Def. X scheme, $\mathcal{F} \in \mathrm{Mod}_{O_X}$ is called globally generated if there is a surjection of O_X mods $\bigoplus_I O_X \longrightarrow \mathcal{F}$, ($|I|$ need not be finite)

Rmk. Note $\mathrm{Hom}_{O_X}(O_X, \mathcal{G}) \xleftarrow{\sim} \Gamma(X, \mathcal{G})$. So giving a sure morphism of O_X-mod $\bigoplus_I O_X \to \mathcal{G}$ is the same as choosing $|I|$ many elts of $\Gamma(X, \mathcal{G})$, such that those generate every stalk $\mathcal{G}_x$, $x \in X$.

Def: An invertible O_X-mod $\mathcal{L}$ is called ample, if for any $\mathcal{F} \in \mathrm{Coh}(X)$ $\exists\, n_{\mathcal{F}} \in \mathbb{N}$ s.t $\forall\, n \geq n_{\mathcal{F}}$,
$\mathcal{F} \otimes_{O_X} \mathcal{L}^n$ is globally generated.

Prop: $\mathcal{L}$ is ample $\iff$ $\mathcal{L}^n$ is ample for some $n \in \mathbb{N}_{>0}$.

Thm: Let $R = \bigoplus_{\lambda \in \mathbb{N}} R_\lambda$ be a noetherian graded ring. $X = \mathrm{Proj}(R)$
If for some $m > 0$, $O_X(m)$ is invertible, $O_X(m)$ is ample.

Pf. Recall R is noeth iff R_0 is noeth and $\exists\ g_1, g_2 \ldots g_r$ homo of +ve deg such that $R = R_0[g_1 \ldots g_r]$

Let $d = \text{lcm}(\deg g_1, \deg g_2, \ldots, \deg g_r)$

Lemma: Let d' be a +ve integer multiple of d. Then the graded subring $\bigoplus_{j=0}^{\infty} R_{d'rj} \overset{\wedge}{\subseteq} R$ is generated as an R_0-algebra by elts of degree $d'r$.

Pf. See

By the Prop before it's enough to show that $\mathcal{O}_X(dmr) \cong \mathcal{O}_X(m)^{\otimes dr}$ is ample. We show this.

Recall $R^{(dmr)}$ is the graded ring whose underlying ring is $\bigoplus_{j=0}^{\infty} R_{dmrj}$ but whose λ-th graded piece is $R_{dmr\lambda}$. By the lemma above $R^{(mdr)}$ is generated over it's degree 0-piece - R_0, by it's set of degree 1 elts. So $R^{(mdr)}$ is a standard graded ring.

The above modifications with the following Prop often allows to reduce a problem about non-standard graded ring to a standard graded ring.

Prop: Let S be an $\mathbb{N}$-graded ring, $a \in \mathbb{N}$, the inclusion map of rings $S^{(a)} \hookrightarrow S$ (it takes deg j elts to deg aj elts) induces an isom φ: $\text{Proj } S \longrightarrow \text{Proj}(S^{(a)})$ such that $\varphi^*(\mathcal{O}_{S^{(a)}}(1)) \cong \mathcal{O}_S(a)$

Pf: Ex.

Returning to the proof of ampleness, take $a = mdr$.

Consider $R^{(a)} \hookrightarrow R$.

Claim: If S is a standard graded noeth ring. $\mathcal{O}_{\text{Proj}(S)}(1)$ is ample.

Pf. Given a coh sheaf $\mathcal{G}$, choose a finitely generated S mod N such that $\mathcal{F} \cong \tilde{N}$. Choose homo elts $n_1, n_2, \ldots, n_s$ of N that generates N. Permuting the order assume $\deg n_1 \le \deg n_2 \le \ldots \le \deg n_s$

$n_1, n_2 \ldots, n_s$ of N that generates N. Permuting the order assume $\deg n_1 \le \deg n_2 \le \cdots \le \deg n_s$.

Claim For $\lambda \ge \lambda' \ge \deg n_s$, $N_\lambda = S_{\lambda-\lambda'} \cdot N_{\lambda'}$

Pf. For $\lambda \ge \deg n_s$, choose $x \in N_\lambda$.

$x = f_1 n_1 + \cdots + f_r n_r$, $\deg f_i = \deg x - \deg n_i$

So f_i is sum of pdt of $\deg x - \deg n_i$ monomials in the deg 1 generators of S. So each f_i can be written as $f_i = \sum \tilde{f}^i_j \cdot g_j$, where $\deg \tilde{f}^i_j = \deg n_s - \deg n_i$

$\deg g_j = \deg x - \deg n_s$, Thus $x = \sum_i \sum_j (\tilde{f}^i_j \cdot n_i) \cdot g_j$

$\deg \tilde{f}^i_j \cdot n_i = \deg n_s \quad \forall i, j$. So $N_\lambda = S_{\lambda - \deg n_s} \cdot N_{\deg n_s}$

So $N_\lambda = S_{\lambda-\lambda'} \cdot S_{\lambda' - \deg n_s} N_{\deg n_s}$

$= S_{\lambda-\lambda'} \cdot N_{\lambda'}$

Claim: For $\lambda \ge \deg n_s$; $\tilde{N}(\lambda)$ is glob gen.

Pf. Take $\lambda \ge \deg n_s$.

- Since S is noeth, N f.g, $\exists$ homo elts $\alpha_1, \alpha_2 \ldots, \alpha_t$ generating N_λ as an S_0-mod.

Consider the graded map

$$* \quad \bigoplus^t S \longrightarrow N(\lambda)$$
$$1 \longmapsto \alpha_i$$

By the claim above the map is sur onto $\bigoplus_{j \ge \lambda} N_j$.

So the cokernel is annihilated by $(S_+)^{\deg s}$.

So the sheaf given by cokernel ~~is~~ on $\operatorname{Proj} S$ is 0.

Thus taking $\sim$ in $*$, get a sur map

$$\bigoplus^t \mathcal{O}_{\operatorname{Proj}(S)} \longrightarrow \widetilde{N(\lambda)} \longrightarrow 0$$

Claim: Since S is standard graded $\tilde{N}(\lambda) \xleftarrow{\sim} \tilde{N} \otimes \mathcal{O}_{\operatorname{Proj}(S)}(\lambda)$

Pf. Ex.

Recall $a = \dim k$ and $R^{(a)} \hookrightarrow R$ induces an isom

$\varphi : \operatorname{Proj}(R) \longrightarrow \operatorname{Proj}(R^{(a)})$ such that $\varphi^*(\mathcal{O}_{R^{(a)}}(1))$
$= \mathcal{O}_R(a)$

Since φ is an isom, pull back of an ample sheaf
is ample. Thus $\mathcal{O}_R(dm+r)$ is ample $\Rightarrow$ $\mathcal{O}(d)$ is ample.

Divisors

Wednesday, November 13, 2024 9:11 AM

Today X stands for an integral (reduced, irreducible) noetherian scheme.

Def: The group of Weil divisors is the additive free abelian subgp spanned by the irreducible closed subsets of codim 1. It's denoted $\mathrm{Weil}(X)$.

- A member of the group of Weil divisors is called a Weil divisor. So a Weil divisor is a formal sum $\sum_{i=1}^{t} n_i Z_i$, where Z_i irr and $\operatorname{codim}_X(Z_i) = 1$, $n_i \in \mathbb{Z}$

- $\sum_{i=1}^{t} n_i Z_i + \sum_{i=1}^{t} n_i' Z_i = \sum_{i=1}^{t} (n_i + n_i') Z_i$

Eg: In $\mathbb{P}^1_{\mathbb{C}}$, codim 1 pts are closed pts. So a Weil divisor looks like $\sum_i n_i([x_i : y_i])$, such as $[1:0] - [0:1]$.

Eg. In $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[x,y])$ $\quad V(x) - 3V(y)$ is a Weil div.

Eg. Take the rational function $x/y^3 \in \mathbb{C}(x,y)$ on $\mathbb{A}^2_{\mathbb{C}}$. $V(x) - 3V(y)$ reads the zeros and poles of this rational function with order.

For today unless otherwise mentioned we shall assume X is regular in codimension 1. This means for every $\xi \in X$ such that $\dim \mathcal{O}_{X,\xi} = 1$, $\mathcal{O}_{X,\xi}$ is a regular ring.

If Z is irr of codim 1 and η_Z be the generic point of Z (i.e. $\overline{\{\eta_Z\}} = Z$), $\dim \mathcal{O}_{Z,\eta_Z} = 1$.

Indeed, recall for $x \in X$, $\operatorname{codim}_X(\overline{\{x\}}) = \dim \mathcal{O}_{X,x}$.

The codim of a point $x \in X$ is by definition $\dim \mathcal{O}_{X,x} = \operatorname{codim}_X(\overline{\{x\}})$.

§: Divisor of a rational function.

Recall: $K(X) =$ Field of rational function of X

$= \mathcal{O}_{X,\eta}$ where η is the generic pt of X.

$= \{ f \in \mathcal{O}_X(U) \mid U \underset{\text{open}}{\subseteq} X \} \Big/ \begin{array}{l} f \in \mathcal{O}_X(U) \\ \sim \tilde{f} \in \mathcal{O}_X(\tilde{U}) \\ \text{iff } \exists V \subseteq U \cap \tilde{U} \\ \text{s.t } f_{|V} = \tilde{f}_{|V} \end{array}$

Given an $Z \underset{\substack{\text{codim } 1, \text{ irr} \\ \text{closed}}}{\subset} X$, since

The local ring $\mathcal{O}_{X,\eta_Z}$ (η_Z gen pt of Z) is reg of dim 1, the maximal ideal is gen by 1 elt. Choose a gen f. For is reg of dim 1, The maximal ideal is gen by 1 elt. Choose a gen f. For $\alpha \in \mathcal{O}_{X,\eta}$, define

$$\boxed{\mathrm{ord}_Z(\alpha) = \max\{n \mid \alpha \in (f^n)\}}$$

note, $\mathrm{ord}_Z(\alpha)$ does not depend on the choice of the gen f, as any other gen is $f \cdot u$ where u is a unit.

Since $K(X) = \mathrm{Frac}(\mathcal{O}_{X,\eta_Z})$, extend ord_Z to $K(X) \longrightarrow \mathbb{Z}$ by setting $\mathrm{ord}_Z(\alpha/\beta) = \mathrm{ord}_Z(\alpha) - \mathrm{ord}_Z(\beta)$

for $\alpha, \beta \in \mathcal{O}_{X,\eta_Z}$. and $\mathrm{ord}_Z(0) = \infty$.

Prop / Def. For $f \in K(X)^*$,

$$\sum_{\substack{Z \text{ irr of} \\ \text{codim } 1}} \mathrm{ord}_Z(f) \cdot Z \quad \text{is an Weil divisor.}$$

This is called the divisor of f and denoted $\mathrm{div}(f)$.

Pf. Since X is noeth, choose a finite affine open cover $X = \bigcup_{i=1}^{r} U_i$.

For each i, write $f = f_i/g_i$. For an irr closed subset Z of codim 1 such that $Z \cap U_i \neq \phi$ or equivalently s.t the gen pt of Z, $\eta_Z \in U_i$, $\mathrm{ord}_Z(f) \neq 0 \Rightarrow$ either $\mathrm{ord}_Z(f_i) \neq 0$ or $\mathrm{ord}_Z(g_i) \neq 0$.

Now for $h \in \mathcal{O}_X(U_i)$, and a codim 1 pt $p \in \mathrm{Spec}(\mathcal{O}_X(U_i)) = U_i$, $\mathrm{ord}_p(h) \neq 0$
$\iff h$ is not a unit in $\mathcal{O}_X(U_i)_p$
$\iff h \in p\mathcal{O}_X(U_i)_p$
$\iff h \in p$ in $\mathcal{O}_X(U_i)$

Now by Krull's Thm The set $\{q \in \mathrm{Spec}(\mathcal{O}_X(U_i)) \mid h \in q\}$ is finite since each such prime is a min prime of $\mathcal{O}_X(U_i)/h$.

So # of codim 1 pts in $\mathcal{O}_X(U_i)$ s.t $\mathrm{ord}_{\overline{\{\xi\}}}(f_i) \neq 0$ or $\mathrm{ord}_{\overline{\{\xi\}}}(g_i) \neq 0$ is finite.

For an irr closed $Z \subseteq X$, $\mathrm{ord}_Z(f) \neq 0$
$\Rightarrow \exists\, i$ $\mathrm{ord}_{Z \cap U_i}(f_i/g_i) \neq 0$ for some i
$\Rightarrow \sum \mathrm{ord}_Z(f)\, Z$ is a finite sum

Rmk. For a codim 1 irr subset Z and $f \in K(X)$, $\mathrm{ord}_Z(f)$ measures The order of zeros and poles f along Z. Indeed since $K(X) = \mathrm{Frac}(\mathcal{O}_{X,\eta_Z})$ and η_Z is a principal ideal (t).
$f = t^{\mathrm{ord}_Z(f)} \cdot u$ where $u \in \mathcal{O}_{X,\eta_Z}^*$

f along Z.
and $\mathfrak{m}_Z$ is a principal ideal $(\cdot\, t)$.
any $f = t^{\mathrm{ord}_Z(f)} \cdot u$ where $u \in \mathcal{O}_{X,\eta_Z}^{*}$

Proof: • For any irr closed Z of codim 1,
$\mathrm{ord}_Z(fg) = \mathrm{ord}_Z(f) + \mathrm{ord}_Z(g)$.
• Thus $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$.
• $\mathrm{div}(1) = 0$ • $\mathrm{div}(1/f) = -\mathrm{div}(f)$
• Thus $\{\mathrm{div}(f) \mid f \in K(X)^*\} \subseteq \mathrm{Weil}(X)$
is a subgp.

Def: The quotient gp $\dfrac{\mathrm{Weil}(X)}{\{\mathrm{div}(f) \mid f \in K(X)^*\}}$ is called the Weil divisor class gp or simply the divisor class gp and denoted $\mathrm{Cl}(X)$.

Eg. 1) $\mathrm{Cl}(\mathbb{A}^n_k) = \{0\}$ where k is a field.
$\mathbb{A}^n_k = \mathrm{Spec}(k[x_1, \dots, x_n])$. Z irr closed of codim 1 $\Rightarrow \exists$ an irr pol f s.t
$Z = V(f)$. Note $\mathrm{div}(f) = Z$.
So $K(\mathbb{A}^n_k) \xrightarrow{\mathrm{div}} \mathrm{Weil}(\mathbb{A}^n_k)$ is surj.

Eg 2: $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$. Given a codim 1 irr closed Z, $\exists$ a homo irr pol f, such that
$Z = V(f)$, define $\deg Z = \deg f$.
Have a ~~surj~~. $\deg : \mathrm{Weil}(\mathbb{P}^n_k) \to \mathbb{Z}$. $(\deg(V(x_0)) = 1)$

Given $D \in_{\mathbb{Z}} \mathrm{Weil}(\mathbb{P}^n_k)$, write
$$D = \sum_{i=1}^{r} n_i V(f_i) - \sum_{j=1}^{m} m_j V(g_j) \text{ where}$$
$n_i, m_j > 0$.
$$\deg D = \sum_{i=1}^{r} n_i \deg(f_i) - \sum_{j=1}^{m} m_j \deg(g_j) = 0$$
$$\Leftrightarrow \quad \deg\Big(\prod_{i=1}^{r} f_i^{n_i}\Big) = \deg\Big(\prod_{j=1}^{m} g_j^{m_j}\Big)$$
$$\Leftrightarrow \quad h = \frac{\prod^{r} f_i^{n_i}}{\prod_{j=1}^{m} g_j^{m_j}} \in K(\mathbb{P}^n_k) \text{ and}$$
$$D = \mathrm{div}(h).$$

So $\deg$ induces an isom $\mathrm{Cl}(\mathbb{P}^n_k) \xrightarrow{\sim} \mathbb{Z}$.
$[V(x_0)]$ being a gen.

End of 15.11.24 lecture

Def. Y integral scheme. $\underline{K(Y)}$ is the constant sheaf $\underline{K(Y)}(U) = K(Y)$.

Def. Given $D \in \mathrm{Weil}(X)$, define the sheaf $\mathcal{O}_X(D) \subseteq \underline{K(X)}$
$\dots \{\dots\} \cup \{0\} \subseteq \underline{K(X)}(U) = K(X)$

sheaf $K(X)$ ...

Def. Given $D \in \mathrm{Weil}(X)$, define the sheaf $\mathcal{O}_X(D) \subseteq K(X)$ by $\mathcal{O}_X(D)(U) = \{f \in K(X)^* \mid (\mathrm{div}(f) + D)|_U \geq 0\} \cup \{0\} \subseteq K(X)(U) = K(X)$

where for $D' = \sum_i n_i Z_i \in \mathrm{Div}(X)$,

- $D'|_U = \sum_i n_i (Z_i \cap U) \in \mathrm{Div}(U)$; • $D' \geq 0 \overset{\text{def}}{\iff} n_i \geq 0 \ \forall i$.

Prop. $\mathcal{O}_X(D) \subseteq K(X)$ is an additive subgp sheaf

Pf. It is straightforward to check $\mathcal{O}_X(D)$ is a sheaf.

We verify $\mathcal{O}_X(D)(U) \subseteq K(X)$ is a subgp.

Let $f_1, f_2 \in \mathcal{O}_X(D)(U)$ and $Z \underset{\text{codim } 1}{\subseteq} X$ such that $Z \cap U \neq \emptyset$. Choose a generator of the maximal ideal $\mathcal{O}_{X,\eta_Z}$, denoted t.

$t^{\mathrm{ord}_Z(f_1)} \mid f_1$, $t^{\mathrm{ord}_Z(f_2)} \mid f_2$ in $\mathcal{O}_{X,\eta_Z}$.

Thus $t^{\min\{\mathrm{ord}_Z(f_1), \mathrm{ord}_Z(f_2)\}} \mid f_1 + f_2$

$\Rightarrow$ $\mathrm{ord}_Z(f_1 + f_2) \geq \min\{\mathrm{ord}_Z(f_1), \mathrm{ord}_Z(f_2)\}$.

$\Rightarrow$ If $(D + \mathrm{div} f_1)|_U \geq 0$, $(D + \mathrm{div}(f_2))|_U \geq 0$. Then $(D + \mathrm{div}(f_1 + f_2))|_U \geq 0 \Rightarrow f_1 + f_2 \in \mathcal{O}_X(U)$

Since $\mathrm{ord}_Z(-f_1) = \mathrm{ord}_Z f_1$, $-f_1 \in \mathcal{O}_X(U)$.

Thus $(\mathcal{O}_X(D)(U), +)$ is a group.

Rmk. Say $D = \sum_{i=1} n_i Z_i$, $\mathrm{div} f = \sum_{i=1} m_i Z_i$, $m_i = \mathrm{ord}_{Z_i}(f)$.

$f \in \mathcal{O}_X(D)(U) \iff$ for each i s.t. $Z_i \cap U \neq \emptyset$. $n_i + \mathrm{ord}_{Z_i}(f) \geq 0$

$\iff$ if $n_i \geq 0$, f can have a pole of order atmost n_i along Z_i, If $n_i \leq 0$, $\mathrm{ord}_{Z_i}(f) \geq -n_i$ i.e f must have a zero of ord atleast n_i along Z_i

Cartier divisors. Y be an integral scheme

Def. • Y integral $K(Y)^*$ is the constant sheaf of abelian gps $K(Y)^*(U) = K(Y)^*$, under pdt.

• $\mathcal{O}_Y^*$ is the sub sheaf of abelian gps given by $\mathcal{O}_Y^*(U) = $ units of the ring $\mathcal{O}_Y(U)$.

• The group (abelian) of Cartier divisors is

$\mathcal{O}_Y(U) = \ldots$

Def. The group (abelian) of Cartier divisors is the group $\Gamma(Y, \underline{K(Y)^*}/\mathcal{O}_Y^*) = \mathrm{Cart}(Y)$

- An elt of $\Gamma(Y, \underline{K(Y)^*}/\mathcal{O}_Y^*)$ is called a Cartier divisor.

- There is a natural ~~map~~ gp homo.

$$K(Y)^* \longrightarrow \mathrm{Cart}(Y)$$

The quotient gp, denoted $\mathrm{CaCl}(Y)$, is called the Cartier divisor class group.

Rmk. Given $s \in \mathrm{Cart}(Y)$, $\exists$ an open covering $Y = \bigcup_{i \in I} U_i$ and $f_i \in K(Y)^*$ s.t $s|_{U_i}$ is represented by f_i. On $U_i \cap U_j$, $f_i = u_{ij} f_j$ for some $u_{ij} \in \mathcal{O}_Y(U_i \cap U_j)^*$

$\boxed{\text{* } X \text{ noeth, integral, reg in codim } 1.}$

Def/Prop: Given Z irr closed of codim 1,

$$\left(\underline{K(X)^*}/\mathcal{O}_X^*\right)_{\eta_Z} \cong \frac{K(X)^*_{\eta_Z}}{(\mathcal{O}_X^*)_{\eta_Z}} = \frac{K(X)^*}{\mathcal{O}^*_{X,\eta_Z}}$$

For $h \in \mathcal{O}^*_{X,\eta_Z}$, $\mathrm{ord}_Z(h) = 0$.

Thus $\mathrm{ord}_Z : K(X)^* \longrightarrow \mathbb{Z}$ factors through $K(X)^*/\mathcal{O}^*_{X,\eta_Z}$.

- Given $s \in \mathrm{Cart}(X)$, define $\boxed{\mathrm{ord}_Z(s) = \mathrm{ord}_Z(s_{\eta_Z} \in \frac{K(X)^*}{\mathcal{O}^*_{X,\eta_Z}})}$

- Given $s \in \mathrm{Cart}(X)$, define

$$\boxed{\mathrm{div}(s) = \sum \mathrm{ord}_Z(s) \cdot Z \in \mathrm{Weil}(X).}$$

- The following claim gives a way to 'think about' the sum in $\mathrm{div}(s)$ for $s \in \mathrm{Cart}(X)$ and justifies why the sum on the right hand is finite.

Claim:* Given $s \in \mathrm{Cart}(X)$, choose a finite open covering $X = \bigcup_{i=1}^{r} U_i$ such that for each i, $\exists f_i \in K(X)^*$ satisfying $s|_{U_i} = f_i$ in $\underline{K(X)^*}/\mathcal{O}_X^*(U_i)$

Then $\mathrm{div}(s)|_{U_i} = \mathrm{div}(f_i)|_{U_i}$

... such that $Z \cap U_i \neq \emptyset$.

Then $\operatorname{div}(s)|_{U_i} = \operatorname{div}(f_i)|_{U_i}$

Pf. For $Z \subseteq X$ irr closed codim 1 such that $Z \cap U_i \neq \emptyset$.
$\operatorname{ord}_Z(s) = \operatorname{ord}_Z(f_i)$. So the claim follows.

Prop. (i) $\operatorname{div}: \operatorname{Cart}(X) \to \operatorname{Weil}(X)$ is a gp homo.

(ii) For $f \in K(X)^*$, $\operatorname{div}(f \in \operatorname{Cart}(X)) = \operatorname{div}(f \in K(X)^*)$

So div induces a gp homo.

$\overline{\operatorname{div}}: \operatorname{CaCl}(X) := \dfrac{\operatorname{Cart}(X)}{K(X)^*} \longrightarrow \operatorname{Cl}(X)$

(iii) When X is normal, div and $\overline{\operatorname{div}}$ are injective.

Def. • A Weil divisor D on X is called locally principal if $\exists$ a (finite) open covering $X = \cup U_i$ and $f_i \in K(X)^*$ such that $D|_{U_i} = \operatorname{div} f_i|_{U_i}$ for each i.

• A Weil divisor D is called effective if $D = \sum_{i=1}^{r} n_i Z_i$ where $n_i \geq 0$

Prop 1) $\operatorname{div}(\operatorname{Cart}(X)) \subseteq$ locally principal Weil divisors.

2) When X is normal, the above containment is an equality.

Pf. 1) follows from claim $*$.

2) Given a locally principal Weil divisor D, choose a finite open covering $X = \bigcup_{i=1}^{r} U_i$ and $f_i \in K(X)^*$ such that $D|_{U_i} = \operatorname{div} f_i|_{U_i}$. Since $\operatorname{div} f_i|_{U_i \cap U_j} = \operatorname{div}(f_j)|_{U_i \cap U_j}$
$= D|_{U_i \cap U_j}$

$\operatorname{div}(f_i/f_j)|_{U_i \cap U_j} = 0$. Since X is normal
$f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^*$. So $\{f_i \in K(X)^*/\mathcal{O}_X^*(U_i)\}_i$
glue to give a global section

$\Gamma / f_j \in O_X(U_i \cap U_j)$ ... $/O_X$ glue to give a global section
$s \in \Gamma(X, \underline{K(X)^*}/O_X^*)$.

By claim $*$, $\operatorname{div}(s) = D$.

- We examine surjectivity of $\operatorname{Cart}(X) \xrightarrow{\operatorname{div}} \operatorname{Weil}(X)$.

The following Thm is crucial.

<u>Thm.</u> Let A be a noetherian ring. A is a unique factorization domain $\iff$ A is normal and $\operatorname{Cl}(\operatorname{Spec} A) = \{0\}$.

<u>Pf.</u> See Prop 6.2, Hart.

<u>Thm.</u> Let X be a normal, noetherian scheme
$\operatorname{Cart}(X) \xrightarrow{\operatorname{div}} \operatorname{Weil}(X)$ is surjective (eq. bijective)
$\iff$ X is <u>locally factorial</u>, i.e for every affine $U \underset{\text{open}}{\subseteq} X$
$O_X(U)$ is a U.F.D
$\iff$ $\overline{\operatorname{div}} : \operatorname{CaCl}(X) \xrightarrow{\sim} \operatorname{Cl}(X)$.

<u>Pf.</u> We only verify $\Leftarrow$ of the top $\iff$.
We know $\operatorname{div}(\operatorname{Cart}(X)) =$ locally principal Weil divisors of X.

Given $D \in \operatorname{Weil}(X)$ and $U \underset{\text{affine}}{\subseteq} X$ open. $D|_U \in \operatorname{Weil}(U)$. But the Thm above $\operatorname{Cl}(U) = \{0\}$, $\exists f \in K(U)^* = K(X)^*$ such that $D|_U = \operatorname{div} f|_U$.
So D is locally principal.

End of 20.11.24 Lecture

Start of 22.11.24 Lecture

<u>Thm.</u> When X is regular, $\operatorname{div}: \operatorname{Cart}(X) \xrightarrow{\sim} \operatorname{Weil}(X)$
and $\overline{\operatorname{div}} : \operatorname{CaCl}(X) \xrightarrow{\sim} \operatorname{Cl}(X)$.

§ Picard group and CaCl

The results of this section are true for any integral scheme, which are not necessarily locally integral

The ... integral scheme, which are not necessarily everywhere smooth.

Thm. Y be an integral scheme. Given a $\delta \in \mathrm{Cart}(Y)$, define
$$\mathcal{L}(\delta)(U) = \{ f \in K(Y)^* \mid f_x \cdot \delta_x \in \mathcal{O}_x(U) - \{0\} / \mathcal{O}^*_{Y,x} \} \cup \{0\} \quad \forall x \in U$$
$$\subseteq \underline{K(Y)}(U) = K(Y)$$

Then 1) $\mathcal{L}(\delta)$ is a sub $\mathcal{O}_Y$-mod sheaf of the sheaf of abelian gps $(\underline{K(Y)}, +)$.

2) $\mathcal{L}(\delta)$ is invertible

Pf 1) is straightforward.

2) Claim: Given $\delta \in \mathrm{Cart}(Y)$. Choose a covering $Y = \bigcup_{i \in I} U_i$ s.t. $\delta|_{U_i} = f_i$ in $\Gamma(U_i, K(Y)^*/\mathcal{O}^*_Y)$

Then $\mathcal{L}(\delta)|_{U_i} \xleftarrow{\sim} \mathcal{O}_{U_i}$.
$$1/f_i \longleftarrow 1$$

In particular $\mathcal{L}(\delta)(U_i) = 1/f_i \, \mathcal{O}_Y(U_i)$ and $\mathcal{L}(\delta)|_{U_i} \xrightarrow{\sim} 1/f_i \, \widetilde{\mathcal{O}_Y(U_i)}$ when U_i is affine.

Pf- $1/f_i \in \Gamma(U_i, \mathcal{L}(\delta))$ as $f_i \cdot 1/f_i \in \mathcal{O}^*_{Y,y}$ $\forall y \in U_i$

$$\begin{array}{ccc} \mathcal{L}(\delta)|_{U_i} & \xrightarrow{\cdot f_i} & \mathcal{O}_{U_i} \\ \cap & & \cap \\ \underline{K(Y)} & \xrightarrow[f_i]{} & \underline{K(Y)} \end{array}$$
gives the inverse.

Thm. Let Y be an integral scheme.

$\mathcal{L}(\delta)$ gives a bijection

Thm. Let Y be an integral scheme.

1) The map $s \longmapsto \mathcal{L}(s)$ gives a bijection
$$\mathrm{Cart}(Y) \longleftrightarrow \left\{ \begin{array}{l} \text{invertible subsheaves} \\ \text{of } \underline{\mathcal{K}(Y)} \end{array} \right\}.$$

2) The map in (1) induces an isom of groups
$$\mathrm{CaCl}(Y) \xrightarrow{\sim} \left\{ \begin{array}{l} \text{isom classes of invertible} \\ \text{subsheaves of } \underline{\mathcal{K}(Y)} \end{array} \right\}.$$
where the group structure on the right again comes from $\otimes\, \mathcal{O}_Y$

3) The map induced by (2)
$$\begin{tikzcd} \mathrm{CaCl}(Y) \arrow[r, "\sim"', "(2)"] \arrow[dr] & \left\{ \begin{array}{l} \text{isom classes of invertible} \\ \text{subsheaves of } \underline{\mathcal{K}(Y)} \end{array} \right\} \arrow[d] \\ & \mathrm{Pic}(Y) \end{tikzcd}$$
is an isom. of groups.

Pf. 1) Injection: Assume $s_\alpha, s_\beta \in \mathrm{Cart}(Y)$ have the same image. Choose an open covering
$$Y = \bigcup_{i \in I} U_i$$
such that $\exists\ f_i^\alpha, f_i^\beta$ for each i satisfying, $s_\alpha|_{U_i} = f_i^\alpha$, $s_\beta|_{U_i} = f_i^\beta$ $\forall i$.

Thus $\frac{1}{f_i^\alpha} \mathcal{O}_Y(U_i) = \frac{1}{f_i^\beta} \mathcal{O}_Y(U_i)$ $\forall i$

$\Rightarrow f_i^\beta / f_i^\alpha,\ f_i^\alpha / f_i^\beta \in \mathcal{O}_Y(U_i)$ $\forall i$

$\Rightarrow f_i^\beta / f_i^\alpha \in \mathcal{O}_Y(U_i)^*$ $\forall i$

$\Rightarrow s_\alpha = s_\beta \in \Gamma(Y, \underline{\mathcal{K}(Y)^*} / \mathcal{O}_Y^*)$.

2) Surjectivity: Given an invertible $\mathcal{O}_Y$ submod $\mathcal{F}$ of

Surjectivity: Given an invertible $\mathcal{O}_Y$ submod $\mathcal{F}$ of $\underline{K(Y)}$, choose a open covering $Y = \bigcup_{i \in I} U_i$

such that for each i $\mathcal{F}|_{U_i} = f_i \cdot \mathcal{O}_{U_i}$ for some $f_i \in K(Y)^*$

Since $f_i \cdot \mathcal{O}_{U_i}|_{U_i \cap U_j} = f_j\, \mathcal{O}_{U_j}|_{U_i \cap U_j}$

$f_i / f_j \in \mathcal{O}_{U_i}(U_i \cap U_j)^*$

$\Rightarrow \{ 1/f_i \in \Gamma(U_i, \underline{K(Y)^*}/\mathcal{O}_Y^*)\}_{i \in I}$ glue

to give a section $s \in \Gamma(Y, \underline{K(Y)^*}/\mathcal{O}_Y^*)$

By Claim $**$, $\mathcal{L}(s) = \mathcal{F}$.

2) By claim $**$, we have

$$\mathcal{L}(s_1) \otimes_{\mathcal{O}_Y} \mathcal{L}(s_2) \xrightarrow{\sim} \mathcal{L}(s_1 s_2)$$

So we only need to check: if $\mathcal{L}(s) \xrightarrow{\sim} \mathcal{O}_Y$, Then $\exists\, f \in K(Y)^*$ such that $s = f$ in $\Gamma(Y, \underline{K(Y)^*}/\mathcal{O}_Y^*)$.

To that end, fix an isom

$$\mathcal{O}_Y \xrightarrow{\sim} \mathcal{L}(s).$$

Denote the image of $1 \in \mathcal{O}_Y(Y)$ via this isom by $g \in K(Y)^*$.

So $\mathcal{L}(s)(U) = g \cdot \mathcal{O}_Y(U) \quad \forall\, U \subseteq_{\text{open}} Y$.

Choose an open cover $Y = \bigcup_{i \in I} U_i$ s.t $s|_{U_i} = f_i$ for some $f_i \in K(Y)^*$.

By claim $**$, $\mathcal{L}(s)(U_i) = 1/f_i\, \mathcal{O}_Y(U_i)$

Thus $1/f_i\, \mathcal{O}_Y(U_i) = g \cdot \mathcal{O}_Y(U_i) \quad \forall i$

$1/f_i = g x$

$g = 1/f_i \cdot y$

$\Rightarrow g / (1/f_i) \in \mathcal{O}_Y(U_i)^* \quad \forall i$

$\Rightarrow s = 1/g \in \Gamma(Y, \underline{K(Y)^*}/\mathcal{O}_Y^*)$ $\square$.

... once we show every invertible ...

3) follows once we show every invertible $\mathcal{O}_Y$-mod is isom to some invertible sub $\mathcal{O}_Y$-mod of $\underline{K(Y)}$. Given an invertible $\mathcal{O}_Y$-mod $\mathcal{L}$, we produce an injective $\mathcal{O}_Y$-mod map $\mathcal{L} \to \underline{K(Y)}$. The image will be the desired invertible $\mathcal{O}_Y$-submod isom to $\mathcal{L}$.

To that end, choose an isom

$$\mathcal{L}_\eta \xrightarrow[\varphi]{\sim} K(Y)$$

where η is the gen pt of Y.

For $U \underset{\text{open}}{\subseteq} Y$, the canonical map composed with φ

$$\mathcal{L}(U) \longrightarrow \mathcal{L}_\eta \xrightarrow[\varphi]{\sim} K(Y)$$

gives a map of

$$\psi_U : \mathcal{L}(U) \longrightarrow \mathcal{L}_\eta \xrightarrow{\sim} \underline{K(Y)}(U) = K(Y)$$

ψ_U induces a $\mathcal{O}_Y$-lin map $\mathcal{L} \longrightarrow \underline{K(Y)}$.

Since Y is integral, $\mathcal{L}(U) \hookrightarrow \mathcal{L}_\eta$ and hence ψ_U is inj $\forall U$. $\square$.

Morphism to projective space

Friday, November 22, 2024 10:47 AM

We are interested (a general version) of the following question.

Question. Let k be a field. X be a finite type scheme $/k$. When is X quasi-projective, i.e. when is X isom to an open subscheme of a closed subscheme of $\mathbb{P}^n_k$.

Notation and Terminology:

$\mathbb{P}^n_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[x_0, \dots, x_n])$, For any scheme Y

$\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} Y$, $\mathcal{O}_{\mathbb{P}^n_Y}(1) = j^* \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$.

$$\begin{tikzcd} \mathbb{P}^n_Y \arrow[r, "j"] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ Y \arrow[r] & \mathrm{Spec}\,\mathbb{Z} \end{tikzcd}$$

$\mathbb{P}^n_{\mathrm{Spec}\,A} =: \mathbb{P}^n_A$,

$\mathcal{L}$ invertible sheaf on X, $s \in \Gamma(X, \mathcal{L})$.

$D_s = \{ x \in X \mid s_x \notin m_x \mathcal{L}_x \}$. $D_s \subseteq X$ open.

Note. The map
$$\mathcal{O}_X \longrightarrow \mathcal{L}$$
$$1 \longmapsto s$$
is an isom on D_s.

Def. For any $t \in \Gamma(V, \mathcal{L})$, $V \supseteq D_s$, $t/s \in \mathcal{O}_X(U)$ is the unique elt s.t $t_{|U} = t/s \cdot s_{|U}$

§ Morphism to $\mathbb{P}^n$

Let A be a ring.

Thm. Let X be an A-scheme (i.e fix a morphism $X \to \mathrm{Spec}\,A$) $\mathcal{L}$ be an invertible $\mathcal{O}_X$-mod. $s_0, \dots s_n \in \Gamma(X, \mathcal{L})$ s.t $X = \bigcup_{i=0}^n D_{s_i}$.

- There is a A-scheme morphism
$$\varphi : X \longrightarrow \mathbb{P}^n_A$$
and an isom $\theta : \varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \xrightarrow{\sim} \mathcal{L}$ which sends $\varphi^*(x_i)$ to s_i

- Any A-scheme morphism
$$\psi : X \longrightarrow \mathbb{P}^n_A$$
such that there is an isom $\psi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \longrightarrow \mathcal{L}$ sending $\psi^*(x_i)$ to s_i must coincide with φ.

Rmk. Given $f : X_1 \longrightarrow X_2$ scheme map, $\mathcal{F}_2 \in \mathrm{Mod}_{\mathcal{O}_{X_2}}$
$t \in \Gamma(X_2, \mathcal{F}_2)$. Recall $f^*\mathcal{F}_2 := f^{-1}\mathcal{F}_2 \otimes \mathcal{O}_{X_1}$

Rmk. Given $f: X_1 \to X_2$ scheme map, $\mathcal{F}_2 \in \mathrm{Mod}_{\mathcal{O}_{X_2}}$, $t \in \Gamma(X_2, \mathcal{F}_2)$. Recall $f^*\mathcal{F}_2 := f^{-1}\mathcal{F}_2 \otimes \mathcal{O}_{X_1}$

t gives a section $f^{-1}(t) \in \Gamma(X_1, f^{-1}(\mathcal{F}_2))$

$f^*(t)$ denotes the section $f^{-1}(t) \otimes 1$.

More canonically, the adjunction between f^*, f_* gives a map (image of $\mathrm{id} \in \mathrm{Hom}(f^*\mathcal{F}_2, f^*\mathcal{F}_2)$), $\mathcal{F}_2 \to f_*(f^*\mathcal{F}_2)$.

f^*t is the image of t under this map.

Pf of Thm.

$$\mathbb{P}^n_A = \bigcup_{i=0}^n D_{x_i}, \quad D_{x_i} = D_+(x_i) = \mathrm{Spec}\,(A[\tfrac{x_0}{x_i}, \cdots \underset{\text{omitted}}{\tfrac{x_i}{x_i}}, \cdots \tfrac{x_n}{x_i}])$$

There is a unique A-scheme map φ_i for each i

$$X \supseteq D_{s_i} \longrightarrow D_{x_i}$$

induced by the A-alg map

$$A[\tfrac{x_0}{x_i} \cdots \widehat{x_i/x_i}, \cdots x_n/x_i] \longrightarrow \Gamma(D_{s_i}, \mathcal{O}_X)$$

$$x_j/x_i \longmapsto s_j/s_i$$

$\varphi_i|_{D_{s_i} \cap D_{s_j}}, \varphi_j|_{D_{s_i} \cap D_{s_j}} : D_{s_i} \cap D_{s_j} \longrightarrow D_{x_i} \cap D_{x_j}$

both correspond to the same A-alg map

$$A[\tfrac{x_0}{x_i}, x_1/x_i, \cdots, x_i/x_i \cdots x_n/x_i][(x_j/x_i)^{-1}] \longrightarrow \Gamma(D_{s_i} \cap D_{s_j}, \mathcal{O}_X)$$

$$\|$$

$$A[\tfrac{x_0}{x_j}, \cdots \quad , x_n/x_j][(\tfrac{x_i}{x_j})^{-1}]$$

- Thus $\varphi_i|_{D_{s_i} \cap D_j} = \varphi_j|_{D_{s_i} \cap D_j}$
- Thus $\exists!\ \varphi: X \to \mathbb{P}^n_A$ such that $\varphi|_{D_{s_i}} = \varphi_i$.

The desired isom $\varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \to \mathcal{L}$ is constructed as follows. Note $\varphi^{-1}(D_{x_i}) = D_{s_i}$ by construction.

$$\theta_i : \varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1))|_{D_{s_i}} \xrightarrow{\sim} \mathcal{O}_{D_{s_i}} \cdot \varphi^*(x_i) \longrightarrow \mathcal{O}_{D_{s_i}} \cdot s_i \xleftarrow{\sim} \mathcal{L}|_{D_{s_i}}$$

$$\varphi^*(x_i)|_{D_{s_i}} \longmapsto s_i|_{D_{s_i}}$$

$$\downarrow\wr \qquad\qquad \uparrow\wr$$

$$\varphi_i^*(\mathcal{O}_{\mathbb{P}^n_A}(1)|_{D_{x_i}}) \xrightarrow{\sim} \varphi_i^*(\mathcal{O}_{\mathbb{P}^n_A}|_{D_{x_i}} \cdot x_i)$$

Since $\theta_i|_{D_{s_i} \cap D_{s_j}} = \theta_j|_{D_{s_i} \cap D_j}$

$\exists!\ \theta: \varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \longrightarrow \mathcal{L}$ s.t. $\theta|_{D_{s_i}} = \theta_i$

Clearly $\theta(\varphi^*(x_i)) = s_i \ \forall i$, $(\theta_j(\varphi^*(x_i)) = \theta_j(\tfrac{s_i}{s_j}\varphi^*(x_j))$

$= s_i/s_j \cdot s_j = s_i$

For the uniqueness, given such a ψ,

The existence of the isom

$$\psi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \xrightarrow{\sim} \mathcal{L} \text{ sending } \psi^*(x_i) \text{ to } s_i$$

implies $\psi^{-1}(D_{x_i}) = D_{s_i}$. Indeed

The isom gives $\psi^*(x_i) \in m_x \psi^*(\mathcal{O}_{\mathbb{P}^n}(1))_x$

$$= \mathcal{O}_{\mathbb{P}^n_A}(1)_{\psi(x)} \otimes m_x$$

The isom gives $\psi^*(x_i) \in m_x \psi^*(\mathcal{O}_{\mathbb{P}^n}(1))_x$
$$= \mathcal{O}_{\mathbb{P}^n_A}(1)_{\psi(x)} \otimes m_x$$
$$\iff s_i \in m_x \mathcal{L}_x.$$
But $\psi^*(x_i) = x_i \otimes 1 \in \mathcal{O}_{\mathbb{P}^n_A}(1)_{\psi(x)} \otimes m_x$
$$\iff x_i \in \mathcal{O}_{\mathbb{P}^n_A}(1)_{\psi(x)}$$ [ψ induces an injection $\frac{\mathcal{O}_{\mathbb{P}^n_A, \psi(x)}}{m_{\psi(x)}} \to \frac{\mathcal{O}_{X,x}}{m_x}$]

$\psi^{-1}(m_x) = m_{\psi(x)}$

Again the existence of the isom $\Rightarrow$

That map $A[x_0/x_i, \dots, x_n/x_i] \to \Gamma(D_{x_i}, \mathcal{O}_X)$

That induces $\psi|_{D_{x_i}}$ sends x_j/x_i to s_j/s_i:

On D_{x_i} $\psi^*(x_j) = \psi^\#(x_j/x_i) \cdot \psi^*(x_i)$

$$\text{So } \psi^\#(x_j/x_i) = \psi^*(x_j)/\psi^*(x_i)$$
$$= s_j/s_i$$

Thus $\psi|_{D_{s_i}} = \varphi|_{D_{s_i}}$ $\forall i$. $\blacksquare$ End of 22.11.24 lecture

Rmk When X is f.t over an algebraically closed field. The map induced by $s_0 \dots s_n$ sends a closed pt $x \in X$ to $[s_0(x) : s_1(x) : \quad : s_n(x)]$ $\in \mathbb{P}^n_k$.

Thm Given an A-scheme X and $n \in \mathbb{N}$. There is a one to one correspondence

$$\frac{\left\{ \mathcal{L}, \text{ ordered tuple } (s_0, \dots, s_n) \;\middle|\; \begin{array}{l} \mathcal{L} \text{ invertible} \\ s_i \in \Gamma(X, \mathcal{L}) \text{ s.t} \\ \bigcup_{i=0}^n D_{s_i} = X \end{array} \right\}}{\sim} \longrightarrow \left\{ \begin{array}{c} A\text{-scheme map} \\ X \to \mathbb{P}^n_A \end{array} \right\}$$

where $(\mathcal{L}_1, (s_0 \dots s_n)) \sim (\mathcal{L}_2, (t_0 \dots t_n))$ iff $\exists$ an isom of $\mathcal{O}_X$-mods $\theta : \mathcal{L}_1 \to \mathcal{L}_2$ sending s_i to t_i $\forall i$.

Pf. Given $\varphi : X \to \mathbb{P}^n_A$ of A-schemes take $\mathcal{L}_\varphi = \varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1))$ and $s_i = \varphi^*(x_i)$.

The morphism induced by $(\mathcal{L}_\varphi, (s_0 \dots s_n))$ is indeed φ. The one-one correspondence is H.W.

Eg: $k = \mathbb{C}, \overline{\mathbb{Q}}, \dots$ $X = \mathbb{P}^1_k$, $\mathcal{L} = \mathcal{O}(2)$
$\Gamma(\mathbb{P}^1, \mathcal{O}(2)) = k x^2 \oplus kxy \oplus k y^2$.

get
$$\mathbb{P}^1_k \longrightarrow \mathbb{P}^2_k$$
$$[x_0 : x_1] \longrightarrow [x_0^2 : x_1 x_2 : x_1^2]$$
The image is $V(Y^2 - XZ)$

§ Criterion for having an embedding

Recall. A locally closed subspace of a topological space is an intersection of a closed and open subset

Recall. A <u>locally closed</u> subspace of a topological space is an intersection of a closed and open subset with the induced topology.

Def. • A morphism of schemes $\varphi: X \to Y$ is called a locally closed immersion if (1) φ induces a homeo from X to a locally closed subset of Y and
(2) $\mathcal{O}_{Y,y} \to (\varphi_* \mathcal{O}_X)_y$ is surjective.
at all points of $y \in \varphi(X)$

Locally closed immersion

• $\varphi: X \to Y$ is called a <u>closed</u> <u>immersion</u> if $\varphi(X) \subseteq Y$ is closed.

• $\varphi: X \to Y$ is called an open immersion if $\varphi(X) \subseteq Y$ is open and $\forall\, y \in \varphi(X)$, $\mathcal{O}_{Y,y} \to (\varphi_* \mathcal{O}_X)_y$ is an isom.

Prop Every locally closed immersion $\varphi: X \to Y$ can be factored as
1) $X \xrightarrow{i} U \xrightarrow{j} Y$ where j is an open immersion and i is a closed immersion.

Pf. Write $\varphi(X) = Z \cap U$ where $Z \underset{\text{closed}}{\subseteq} Y$, $U \underset{\text{open}}{\subseteq} Y$.
Then φ factors as $X \xrightarrow{i} U \xrightarrow{j} Z$ as a scheme map, where U has the induced open scheme structure and j is the open immersion.
We claim that i is a closed embedding. Indeed i is a homeo onto the image and $i(X) = Z \cap U$ is closed in U.
Note that for a point in U which is not in $i(X)$, the stalk $(i_* \mathcal{O}_X)_y = \varphi_* (\mathcal{O}_X)_y = 0$. Indeed one can choose an open nbhd V of y such that $V \cap i(X) = \phi$. Then $i^{-1}(V) = \phi$. So $(i_* \mathcal{O}_X)_y = 0$.
We claim that the induced map
$$\mathcal{O}_U \to i_* \mathcal{O}_X \text{ is surjective.}$$
If $y \in i(X)$, $\mathcal{O}_{U,y} \to (i_* \mathcal{O}_X)_y$ is the same as the induced map $\mathcal{O}_{X,y} \to (\varphi_* \mathcal{O}_X)_y$ which is surjective by our assumption.
If $y \notin i(X)$, $\mathcal{O}_{U,y} \to (i_* \mathcal{O}_X)_y$ is surjective because the target is zero.

Rmk. In general it is not true that any locally closed immersion $\varphi: X \to Y$ can be factored as
$X \xrightarrow{j} Z \xrightarrow{i} Y$ where j is an open immersion and i is a closed immersion.
However we have the following.

Prop. Let $\varphi: X \to Y$ be a locally closed immersion such that $\varphi_* \mathcal{O}_X$ is quasi-coherent. Then $\varphi: X \to Y$ can be factored as $X \xrightarrow[j]{} Z \xrightarrow[i]{} Y$
where j is an open immersion and i is closed

Pf. Let $\mathcal{I} = \ker(\mathcal{O}_Y \to \varphi_* \mathcal{O}_X)$. For $y \in \varphi(X)$,

Pf. Let $\mathcal{I} = \ker(\mathcal{O}_Y \to \varphi_*\mathcal{O}_X)$. For $y \in \varphi(X)$,
since $\mathcal{O}_{Y,y} \to (\varphi_*\mathcal{O}_X)_y$ is surj, $\mathcal{I}_y \subseteq m_y$.
Since $\varphi_*\mathcal{O}_X$ is q.coh, $\mathcal{I}$ is q.coh.
So $V(\mathcal{I}) = \{y \in Y \mid \mathcal{I}_y \subseteq m_y\}$ is closed in Y.
$V(\mathcal{I}) \supseteq \varphi(X)$. Equip $V(\mathcal{I})$ with the scheme structure given by $\mathcal{O}_Y/\mathcal{I}$, call the scheme Z.

Note $\varphi(X)$ is open in $V(\mathcal{I})$. Since $\mathcal{I}\cdot\varphi_*\mathcal{O}_X = 0$, the map $\varphi : X \to Y$ factors through the closed immersion $\varphi : X \xrightarrow[j]{} Z \xrightarrow[i]{} Y$, i is the closed immersion.

We claim j is an open immersion. Indeed what remains to check is that $\forall\, y \in \varphi(X) \subseteq Z$, the induced map $\mathcal{O}_{Z,y} \to (j_*\mathcal{O}_X)_y$ is an isom. But
$\mathcal{O}_{Z,y} = \dfrac{\mathcal{O}_{Y,y}}{\mathcal{I}_y}$ is isom to $(j_*\mathcal{O}_X)_y$ for $y \in \varphi(X)$ via the map induced by φ and hence by j. $\square$

Def. Given a scheme Y. Sch_Y is the category
- whose objects are schemes X together with a morphism $X \to Y$
- Given $X_1 \to Y$, $X_2 \to Y$ in Sch_Y, a Y-scheme morphism is a scheme map $\varphi : X_1 \to X_2$ s.t.

$$\begin{tikzcd} X_1 \arrow[rr, "\varphi"] \arrow[dr] & & X_2 \arrow[dl] \\ & Y & \end{tikzcd}$$

commutes.

Def. • Let X be a Y scheme. An invertible $\mathcal{O}_X$ mod $\mathcal{L}$ is called Y-__very ample__ if $X \to Y$ can be factored as

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr] & & \mathbb{P}^n_Y \arrow[dl] \\ & Y & \end{tikzcd} \qquad *$$

where φ is locally closed and $\mathcal{L} \simeq \varphi^*(\mathcal{O}_{\mathbb{P}^n_Y}(1))$.

- Whenever the structure map $X \to Y$ has a factorization as in $*$, we say __X is quasi projective over Y__.

Prop. A ring. Let X/A be proper. Any locally closed immersion $\varphi : X \to \mathbb{P}^n_A$ is a closed immersion. Thus X is projective $/A$ $\iff$ X is quasi-projective $/A$.

Pf. $\mathbb{P}^n_A$ is separated, X/A proper. So $\varphi(X)$ is closed in $\mathbb{P}^n_A$. $\square$

- Whenever the structure map $X \to Y$ can be factored as in $*$ with φ being a closed embedding,

• Whenever the structure map $X \to Y$ can be factored as in $*$ with φ being a closed embedding, the morphism $X \to Y$ is called projective.

Thm. Let X be a finite type scheme $/A$ = noetherian; $\mathcal{L}$ be an invertible sheaf on X.
$\mathcal{L}$ is ample $\iff$ $\mathcal{L}^m$ is very ample for some m.

Pf. In our setup a very ample invertible sheaf $\mathcal{L}'$ is ample.
Indeed, choose a locally closed embedding
$$X \underset{\varphi}{\hookrightarrow} \mathbb{P}^n_A \quad \text{s.t} \quad \varphi^* \mathcal{O}_{\mathbb{P}^n}(1) \xrightarrow{\sim} \mathcal{L}'.$$
Factor φ as $X \xrightarrow[j]{} Z \xrightarrow[i]{} \mathbb{P}^n_A$ where j is an open immersion and i is closed. Given $\mathcal{F} \in Coh(X)$ $\exists$ $\mathcal{G} \in Coh(Z)$ such that $j^*\mathcal{G} \xrightarrow{\sim} \mathcal{F}$. Take $\tilde{\mathcal{F}} = i_* \mathcal{G}$.
Then $\varphi^*(\tilde{\mathcal{F}}) \xrightarrow{\sim} j^*(i^*(i_*\mathcal{G})) \xrightarrow{\sim} j^*\mathcal{G} \xrightarrow{\sim} \mathcal{F}$.

Since $\mathcal{O}_{\mathbb{P}^n}(1)$ is ample, $\exists$ m_0 s.t $\forall$ $m \geq m_0$ $\tilde{\mathcal{F}} \otimes \mathcal{O}_{\mathbb{P}^n}(m)$ is globally gen. Then $\varphi^*(\tilde{\mathcal{F}} \otimes \mathcal{O}_{\mathbb{P}^n}(m)) \xrightarrow{\sim} \mathcal{F} \otimes \mathcal{L}'^m$
is also glob gen. So $\mathcal{L}'$ is ample.
$\mathcal{L}^m$ very ample $\Rightarrow$ $\mathcal{L}^m$ ample $\Rightarrow$ $\mathcal{L}$ ample.

Now assume $\mathcal{L}$ is ample.
Step 1: For each $x \in X$, there exists $s_x \in \Gamma(X, \mathcal{L}^{n_x})$ such that D_{s_x} is an affine open nbhd of x.

Pf. of Step 1: Fix $x \in X$ and an affine open nbhd U_x s.t. $\mathcal{L}|_{U_x} \cong \mathcal{O}_{U_x}$ of x. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal sheaf s.t $V(\mathcal{I}) = X - U_x$. Choose n_x s.t $\mathcal{I} \otimes \mathcal{L}^n$ is globally generated for $n \geq n_x$. So there exists $s_x \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{n_x}) \subseteq \Gamma(X, \mathcal{L}^{n_x})$
$s_x \notin m_x(\mathcal{I} \otimes \mathcal{L}^{n_x}) = m_x \mathcal{L}^{n_x}$ [$\because$ $\mathcal{I}_x = \mathcal{O}_{X,x}$]
Note $\forall$ $y \in X - U_x$, $(s_x)_y \in \mathcal{I}_y \otimes \mathcal{L}^{n_x}_y \subseteq m_y \mathcal{L}^{n_x}_y$

So $D_{s_x} \subseteq U_x$. Choose an isom $\mathcal{L}|_{U_x} \xrightarrow{\sim} \mathcal{O}_{U_x}$. Assume s_x goes to f_x
$D_{s_x} = D_{f_x}$ in U_x. So D_{s_x} is affine. ∎

Since X is quasi-compact, there is a finite covering
$$X = \bigcup_{i=1}^{r} D_{s_{x_i}}$$
Replace $\mathcal{L}$ by $\mathcal{L}^{n_{x_1} n_{x_2} \cdots n_{x_r}}$ and s_{x_i} by $s_{x_i}^{n_{x_1} \cdots n_{x_r}/n_{x_i}}$

Then $\exists$ $s_{x_1} \ldots s_{x_r} \in \Gamma(X, \mathcal{L})$ such that
$$X = \bigcup_{i=1}^{r} D_{s_{x_i}} \quad \text{and each } D_{s_{x_i}} \text{ is affine}$$
For each $1 \leq i \leq r$, make a choice
$$\Gamma(D_{s_{x_i}}, X) = A[y_{ij} \mid 1 \leq j \leq n_i]$$
$\exists$ $l \in \mathbb{N}$ s.t $y_{ij} \otimes s_{x_i}^{l}$ extends to a global section t_{ij} of $\Gamma(X, \mathcal{L}^l)$ $\forall$ $1 \leq i \leq r$, $1 \leq j \leq n_i$

$\exists\ l \in \mathbb{N}$ s.t $y_{ij} \otimes s_{x_i}^l$ extends to a global section t_{ij} of $\Gamma(X, \mathcal{L}^l)$, $\forall\ 1 \le i \le r$, $1 \le j \le n_i$

Consider the morphism φ to some $\mathbb{P}^N_A$ given by $\mathcal{L}, \{t_{ij}, s_{x_i}^l\}_{1\le i\le r,\ 1\le j\le n_i}$. $N = n_1+n_2+\cdots+n_r+r-1$

$\mathbb{P}^N = \operatorname{Proj}(A[x_{ij}, y_i \mid 1\le i\le r,\ 1\le j\le n_i])$

Claim: φ is a locally closed immersion

Pf: Since $X = \bigcup_{i=1}^r D_{s_{x_i}^l}$, φ factors through $\bigcup_{i=1}^r D_{y_i} = V$

We show $\varphi : X \longrightarrow V \subseteq \mathbb{P}^N$ is a closed immersion.

For that, enough to show

$$D_{s_{x_i}} = \varphi^{-1}(D_{y_i}) \longrightarrow D_{y_i} \text{ is a closed immersion } \forall i$$

$$\iff \Gamma(D_{y_i}, \mathbb{P}^N) \longrightarrow \Gamma(D_{s_{x_i}}, X) \text{ is surjective } \forall i$$

$$\Gamma(D_{s_{x_i}}, X) = A[y_{ij} \mid 1 \le j \le r]$$

$$\frac{t_{ij}}{s_{x_i}^l} \longmapsto y_{ij} \quad [\because t_{ij} \text{ extends } y_{ij} \otimes s_{x_i}^l].\ \square$$

End of 27.11.24 lecture

Prop: (i) A noetherian ring. $X \in \mathrm{Sch}_A$ is quasi-projective iff X has an ample invertible sheaf.

(ii) If X is proper $/A$. Then X is projective $\iff X$ has an ample invertible sheaf.

§ Examples of projective morphisms. Blow-ups.

- X noetherian scheme, $\mathcal{S} = \bigoplus_{n\in\mathbb{N}} \mathcal{S}_n$ be an $\mathbb{N}$-graded quasi-coherent sheaf of $\mathcal{O}_X$-algebras such that
- Each $\mathcal{S}_n$ is a q.coh $\mathcal{O}_X$-submod of $\mathcal{S}$
- The $\mathcal{O}_X$-alg structure on $\mathcal{S}$, comes from a ring map $\mathcal{O}_X \to \mathcal{S}_0$
- for each affine $\operatorname{Spec}(A) \subseteq X$
  $\mathcal{S}(\operatorname{Spec} A) = (S_0 \oplus S_1 \oplus S_2 \oplus \cdots)^\sim$
  where $\oplus S_i$ is standard graded.

Prop. There exists a scheme denoted $\operatorname{Proj}_X(\mathcal{S})$ with a map $\pi : \operatorname{Proj}_X(\mathcal{S}) \longrightarrow X$ such that for each affine open $\operatorname{Spec}(A) \subseteq X$, (see page 166 Hart).
$\pi^{-1}(U) \xrightarrow{\sim} \operatorname{Proj}(S_0 \oplus S_1 \oplus \cdots \oplus\ )$ where $\mathcal{S}(\operatorname{Spec} A) = (\oplus S_i)^\sim$

- There exists an invertible sheaf denoted $\mathcal{O}(1)$ on $\operatorname{Proj}_X(\mathcal{S})$ s.t $\mathcal{O}(1)|_{\operatorname{Spec}(A)} \xrightarrow{\sim} \widetilde{S(1)}$.
  where $S = \oplus S_i$.

- Given an invertible sheaf $\mathcal{L}$ and $\mathcal{S}$ as above define $\mathcal{S}' = \oplus \mathcal{S}_n \otimes \mathcal{L}^n$
  Then there is a natural isom
  $\varphi : \operatorname{Proj}(\mathcal{S}') \longrightarrow \operatorname{Proj}(\mathcal{S})$ such that

Thm There is a natural isom

$$\varphi : \underline{\mathrm{Proj}}(\mathcal{S}') \longrightarrow \underline{\mathrm{Proj}}(\mathcal{S}) \text{ such that}$$
$$\mathcal{O}(1) \xrightarrow{\sim} \varphi^* \mathcal{O}(1) \otimes \pi^*(\mathcal{L}) \text{ where}$$

$\pi : \underline{\mathrm{Proj}}(\mathcal{S}) \to X$ is the natural projection map.

__eg.__ $\mathcal{E} = \bigoplus^{r+1} \mathcal{O}_X \quad r \in \mathbb{N}$

$\mathcal{S} = \mathrm{Sym}^{\bullet}(\mathcal{E}) = \mathcal{O}_X \oplus \mathrm{Sym}^1 \mathcal{O}_X^{\oplus r+1} \oplus \mathrm{Sym}^2 \cdots$

On $\mathrm{Spec}\, A$. $\mathcal{S}|_{\mathrm{Spec}\, A} \xrightarrow{\sim} A[T_0, \dots, T_r]$

"So" $\underline{\mathrm{Proj}}(\mathcal{S}) = X \times_A \mathbb{P}^n_A$.

__Prop.__ If X has an ample invertible sheaf $\mathcal{L}$. Then $\pi : \underline{\mathrm{Proj}}(\mathcal{S}) \to X$ is projective.

__Pf__ Choose $n \in \mathbb{N}$ s.t. $\mathcal{S}_1 \otimes \mathcal{L}^n$ is glob gen.

take $\mathcal{L}' = \mathcal{L}^n$.

Choose a surjection

$$\bigoplus^{r+1} \mathcal{O}_X \longrightarrow \mathcal{S}_1 \otimes \mathcal{L}'$$

This gives an $\underset{\mathcal{O}_X\text{-lin}}{\text{surjection}}$ $\underline{\mathrm{Sym}}^{\bullet}(\mathcal{O}_X^{r+1}) \longrightarrow \bigoplus \mathcal{S}_n \otimes \mathcal{L}'^n = \mathcal{S}'$

and thus a closed embedding of sch$_X$.

$$\begin{tikzcd} \underline{\mathrm{Proj}}(\mathcal{S}') \arrow[r, hook] \arrow[dr] & \underline{\mathrm{Proj}}(\underline{\mathrm{Sym}}^{\bullet}(\mathcal{O}_X^{r+1})) = X \times \mathbb{P}^r \arrow[d] \\ & X \end{tikzcd}$$

But $\underline{\mathrm{Proj}}(\mathcal{S}') = \underline{\mathrm{Proj}}(\mathcal{S})$.

__Def.__ Given an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$. The blow up of X along $\mathcal{J}$ is the X-scheme

$$\pi : \underline{\mathrm{Proj}}\Big(\bigoplus_{n \in \mathbb{N}} \mathcal{J}^n\Big) \longrightarrow X. \quad \text{set } Bl_{\mathcal{J}}(X) = \underline{\mathrm{Proj}}\Big(\bigoplus_{n \in \mathbb{N}} \mathcal{J}^n\Big)$$

$$\Big(\bigoplus_{n\in\mathbb{N}} \mathcal{J}^n\Big) = (\mathcal{O}_X \oplus \mathcal{J} \oplus \mathcal{J}^2 \oplus \cdots)$$

__Notation:__ Given $f : Y_1 \to Y_2$ and an ideal sheaf $\mathcal{J}_2 \subseteq \mathcal{O}_{Y_2}$

$$f^{-1}(\mathcal{J}_2)\mathcal{O}_{Y_2} = \mathcal{J}_2 \mathcal{O}_{Y_2}$$

is the ideal sheaf obtained as the image of the map $f^*(\mathcal{J}_2) \to \mathcal{O}_{Y_2}$

__Thm.__ (i) Given an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$, consider the blow-up map

$$\pi : Bl_{\mathcal{J}}(X) \longrightarrow X$$

Then $\pi^{-1}(\mathcal{J})\mathcal{O}_{Bl_{\mathcal{J}}(X)} \xrightarrow{\sim} \mathcal{O}(1)$

In particular $\mathcal{J}\, Bl_{\mathcal{J}}(X)$ is invertible.

(ii) $\pi^{-1}(X \setminus V(\mathcal{J})) \longrightarrow X \setminus V(\mathcal{J})$ is an isom

(iii) Let $f : Z \to X$ be a scheme map such that $\mathcal{I}\mathcal{O}_Z$ is invertible. Then f uniquely factors as

$$\begin{tikzcd} Z \arrow[rr, dashed] \arrow[dr, "f"'] & & Bl_{\mathcal{I}}(X) \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

<u>Pf</u> (i) $\pi^{-1}(\mathcal{I})\,\mathcal{O}_{Bl_{\mathcal{I}}(X)} = \mathcal{I}(\mathcal{O}_X \oplus \mathcal{I} \oplus \cdots)^{\sim}$

check on affine opens in X

$= [(\oplus \mathcal{I}^n)(1)]^{\sim}$

$= \mathcal{O}(1)$

(ii) $\mathcal{I}|_{X \setminus V(\mathcal{I})} = \mathcal{O}_X$. (iii) See prop 7.14.

<u>Thm</u> (Resolving indeterminacy via blow-up).
$\mathcal{L}$ be an invertible sheaf on X, $s_0, \dots s_n \in \Gamma(X, \mathcal{L})$
Let $U = \bigcup_{i=0}^{n} D_{s_i}$. Then we have a map

$$\varphi : U \longrightarrow \mathbb{P}^n_A$$

There exists an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ s.t we have a factorization

$$\begin{tikzcd} & Bl_{\mathcal{I}}(X) \arrow[dl] \arrow[dr, dashed, "\tilde{\varphi}"] & \\ X \supseteq U \arrow[rr, "\varphi"'] & & \mathbb{P}^n_A \end{tikzcd}$$

End of 29.11.24 lecture

Kahler differentials

Friday, November 29, 2024 7:34 AM

The goal is to develop a Theory of cotangent and tangent sheaf of a scheme.

Def: Let A (commutative) ring. $\varphi: A \to R$ ring homo. (So R is an A-alg), M be an R mod. An A-linear derivation from R to M is a map $d: R \to M$ s.t

① d is A-linear: i.e. $d(\alpha+\beta) = d(\alpha) + d(\beta)$ and $d(\lambda\alpha) = \lambda\, d(\alpha)$ $\forall\ \alpha, \beta \in R$, $\lambda \in A$.

② $d(\alpha\beta) = \alpha\, d\beta + \beta\, d\alpha$. (Leibniz rule).

The A-mod of $\forall$ A-linear der $A \to R$ is denoted $\mathrm{Der}_A(R, M)$

- Let $(T, \mathcal{O}_T)$ be a ringed space, $\tilde{\mathcal{O}}_T$ be a sheaf of $\mathcal{O}_T$ algbs, $\mathcal{F}$ be an $\tilde{\mathcal{O}}_T$-mod. An $\mathcal{O}_T$-linear map $d: \tilde{\mathcal{O}}_T \to \mathcal{F}$ is called a derivation if for every open $U \subseteq T$ $d_U: \tilde{\mathcal{O}}_T(U) \to \mathcal{F}(U)$ is an $\mathcal{O}_T(U)$-lin derivation.

Rmk. $d: R \to M$ be A-lin derivation. Then $d(\varphi(a)) = 0$ $\forall\ a \in A$. Indeed $d(\varphi(a)) = \varphi(a)\, d(1)$. and $d(1) = d(1\cdot 1) = 1\cdot d(1) + 1\cdot d(1) \Rightarrow d(1) = 0$

Eg.
- $k[x] \to k[x]$, k field.
  $f \longmapsto \frac{d}{dx} f$ is a -lin derivation
- $k[x] \longrightarrow k[x] \xrightarrow{\frac{d}{dx}} [x]$ $\quad d(f) = \frac{d}{dx}(f(x^2))$
  $x \longmapsto x^2$ ring homo $\quad = 2x \cdot \frac{df}{dx}(x^2)$
  is a derivation.

Thm/Df Given $\varphi: A \to R$ as above, There is an R-mod denoted $\Omega_{R/A}$ and an A-linear derivation $d_{R/A}: R \to \Omega_{R/A}$ such That any A-lin derivation $d: R \to M$ factors <u>uniquely</u> as

$$\begin{tikzcd}
R \arrow[r, "d_{R/A}"] \arrow[dr, "d"'] & \Omega_{R/A} \arrow[d, dashed, "\exists!\ \varphi_d"] \\
 & M
\end{tikzcd}$$

where $\varphi_d: \Omega_{R/A} \longrightarrow M$ is <u>R-linear</u>.

- The pair $(\Omega_{R/A}, d_{R/A})$ is unique <u>upto a unique isom.</u>
- $\Omega_{R/A} :=$ module of Kahler differentials of R over A
  $d_{R/A} :=$ The universal derivation of The A-alg R.

Pf. • uniqueness
Given another pair $(\tilde{\Omega}_{R/A}, \tilde{d}_{R/A})$ get a commutative diag

$$\begin{tikzcd} & R \arrow[dl, "d_{R/A}"'] \arrow[dr, "\tilde{d}_{R/A}"] & \\ \Omega_{R/A} \arrow[rr, shift left, "\varphi_{\tilde{d}_{R/A}}"] & & \tilde{\Omega}_{R/A} \arrow[ll, dashed, shift left, "\tilde{\varphi}_{d_{R/A}}"] \end{tikzcd}$$

Now note $d_{R/A} = \tilde{\varphi}\cdot\varphi\cdot d_{R/A}$

By the uniqueness of the factorization $\tilde{\varphi}\cdot\varphi = id$.
Similarly $\varphi\cdot\tilde{\varphi} = id$.

- Existence: (i) Consider the set of symbols $\{\bar{d}(r)\}_{r\in R}$

Let $\Omega' :=$ free R mod with basis $\{\bar{d}(r)\}_{r\in R}$.

Set $N = R$-sub mod spanned by
$$\{\bar{d}(a) \mid a\in A\} \cup \{\bar{d}(r_1+r_2) - \bar{d}(r_1) - \bar{d}(r_2) \mid r_1, r_2\in R\}$$
$$\cup\ \{\bar{d}(r_1 r_2) = r_1 d r_2 + r_2 d r_1\}$$

Set $\Omega_{R/A} = \dfrac{\Omega'}{N}$, define $d_{R/A} : R \to \Omega_{R/A}$ by $d_{R/A}(r) =$ image of $\bar{d}(r)$ in $\bar{\Omega}/N$.

- $d_{R/A}$ is an A-lin derivation.

- Universal property: Given a derivation $d : R\to M$
  Define $\bar{\varphi} : \bar{\Omega} \to M$ by $\bar{\varphi}(dr) = d(r)$
  $\because$ d derivation $d(N)=0$, so get a unique map
  $$\varphi_d : \bar{\Omega}/N = \Omega_{R/A} \to M \text{ s.t.}$$
  $$\varphi_d(\underbrace{dr\,(\text{mod } N)}_{= d_{R/A}(r)}) = d(r).$$

Example. $R = A[x_1 \ldots x_n]$ polynomial ring.

$$\Omega_{R/A} \underset{\text{free mod}}{=} R\,dx_1 \oplus \cdots \oplus R\,dx_n$$

$d_{R/A} : R \to \Omega_{R/A}$ is given by
$$d_{R/A}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Claim: Given a derivation $d : R\to M$
$$d(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d(x_i)$$

- Since both the sides are A-linear enough to check for ~~monom~~ $f = x_1^{a_1}\cdots x_n^{a_n}$.

So d factorizes as
$$\begin{tikzcd} R \arrow[r, "d_{R/A}"] \arrow[dr, "d"'] & \bigoplus R\,dx_i = \Omega_{R/A} \arrow[d, dashed] \\ & M \end{tikzcd} \qquad dx_i \mapsto d(x_i)$$

Prop. Let I be the kernel $R\otimes_A R \to R$

Prop. Let I be the kernel $R \otimes_A R \to R$
$x \otimes y \mapsto xy$

Consider $d_0 : R \to I/I^2$ given by $d_0(r) =$ image of $1 \otimes r - r \otimes 1 \in I/I^2$

Then $(I/I^2, d_0) =$ (module of Kähler differentials, universal derivation).

Pf. Note $R \otimes_A R \to R$ induces an isom

$$R \otimes_A R/I \xrightarrow{\sim} R$$

Since $I \cdot I/I^2 = 0$, I/I^2 is naturally an R-mod:

For $r \in R$, $y \in I/I^2$

$$r \cdot y := (1 \otimes r) \cdot y = (r \otimes 1) \cdot y$$

- Check $d_0 : R \to I/I^2$ is a derivation.
- Now given an A-lin derivation $d : R \to M$ extend d uniquely to an R-lin map

$$\tilde{d} : R \otimes_A R \to M$$
$$\tilde{d}(r_1 \otimes r_2) = r_1 d(r_2)$$

We show $\tilde{d}(I^2) = 0$.

Claim: Consider $R \otimes_A R$ as an R-mod via the map

$$R \to R \otimes_A R$$
$$r \mapsto r \otimes 1 .$$

Then I is gen by $\{1 \otimes r - r \otimes 1 \mid r \in R\}$ as an R-mod.

Pf. Take $\sum x_i \otimes y_i \in I \Rightarrow \sum x_i y_i = 0$

$$\sum x_i \otimes y_i = \sum [x_i (1 \otimes y_i - y_i \otimes 1) + x_i y_i \otimes 1]$$
$$= \sum x_i (1 \otimes y_i - y_i \otimes 1) + (\underbrace{\sum x_i y_i}_{=0}) \otimes 1$$

$\square$.

$\tilde{d}(I^2) = 0$ follows once we show

$$\tilde{d}((1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1)) = 0$$
$$\Leftrightarrow \tilde{d}(1 \otimes xy - y \otimes x - x \otimes y + xy \otimes 1) = 0$$
$$\Leftrightarrow d(xy) - y\,dx - x\,dy + xy\,d(1) = 0$$
$$\Leftrightarrow d(xy) = y\,dx + x\,dy$$

Note by def $\tilde{d}(d_0(r)) = \tilde{d}(1 \otimes r - r \otimes 1)$
$$= dr - r\,d(1)$$
$$= dr$$

So we have a commutative diag

$= dr$

So we have a commutative diag

$$\begin{tikzcd} & I/I^2 \arrow[dr, "\tilde{d}|_{I/I^2}"] & \\ R \arrow[ur, "d_0"] \arrow[rr, "d"'] & & M \end{tikzcd}$$

The uniqueness of the factorization follows from the claim.

<u>Thm.</u> 1) Given an A-alg map $R \xrightarrow[\psi]{} R'$, there exists <u>a unique map</u>

$$d : \Omega_{R/A} \to \Omega_{R'/A} \text{ such that}$$
$$d\psi(d_{R/A}(r)) = d_{R'/A}(\psi(r)).$$

2) For a multiplicatively closed subset $S \subseteq R$ the canonical map $R \xrightarrow{i} S^{-1}R$ induces an isom

$$S^{-1}(\Omega_{R/A}) \xrightarrow[S^{-1}(di)]{\sim} \Omega_{S^{-1}R/A}.$$

So in particular, for a prime ideal p of R.

$$(\Omega_{R/A})_p \xrightarrow{\sim} \Omega_{R_p/A} \xrightarrow[H.W]{\sim} \Omega_{R_p/A_q}$$

where q is the inverse image of p under $A \to R$.

3) Let A be an A_0-alg. The maps $A_0 \xrightarrow[\psi]{} A \xrightarrow[\varphi]{} R$ <u>induces</u> an <u>exact seq</u>

$$\Omega_{A/A_0} \otimes_A R \to \Omega_{R/A_0} \to \Omega_{R/A} \to 0$$
$$d_{R/A_0}(r) \longmapsto d_{R/A}(r)$$
$$d_{A/A_0}(a) \otimes r \longmapsto r.\, d_{R/A_0}(\varphi(a)).$$

4) Given an ideal $J \subseteq R$, there is an exact seq

$$J/J^2 \to \Omega_{R/A} \otimes_R R/J \to \Omega_{R/J/A} \to 0 \text{ (conormal seq)}$$
$$d_{R/A}(r) \otimes (r' \,(\mathrm{mod}\, J)) \longmapsto r' d_{R/J/A}(r (\mathrm{mod}\, J))$$
$$\alpha\, (\mathrm{mod}\, J^2) \longmapsto d_{R/A}(\alpha) \otimes 1$$

5) Let B be an A-alg. The natural map

$$\Omega_{R/A} \otimes_A B \to \Omega_{R\otimes_A B/B} \text{ is an isom of } R\otimes_A B$$
mods
$$d_{R/A}(r) \otimes b \longmapsto b\, d_{R\otimes_A B/B}(r\otimes 1) = d_{R\otimes_A B/B}(r \otimes b)$$

<u>Pf.</u> In (3), (4), the idea is to apply $\mathrm{Hom}_R(-, M)$ and check the resulting sequences are left exact, for $M \in \mathrm{Mod}_R$ (for 3), $M \in \mathrm{Mod}_{R/J}$ (for 4). We first check (4)

Note for $\alpha, \beta \in J$,

$$d_{R/A}(\alpha\beta) = \alpha\, d_{R/A}(\beta) + \beta\, d_{R/A}(\alpha) \in J\,\Omega_{R/A}$$

So $d_{R/A}(\alpha\beta) = 0 \in \Omega_{R/A} \otimes_R R/J$.

Thm The map $J \to \Omega_{R/A} \otimes_R R/J$ sends J^2 to zero.

For $M \in \mathrm{Mod}_{R/J}$ the resulting seq is

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(J/J^2, M) & \leftarrow & \mathrm{Hom}_R(\Omega_{R/A} \otimes_R R/J, M) & \leftarrow & \mathrm{Hom}_{R/J}(\Omega_{R/J/A}, M) & \leftarrow & 0 \\
\| & & \| & & \| & & \\
\mathrm{Hom}_R(J/J^2, M) & \leftarrow & \mathrm{Der}_A(R, M) & \leftarrow & \mathrm{Der}_A(R/J, M) & \leftarrow & 0
\end{array}$$

Since $R \to R/J$ is surj, the injectivity is clear.
To check the exactness in the middle, take $d \in \mathrm{Der}_A(R, M)$ s.t. $d(J) = 0$.
Then the induced A-lin map $\bar{d} : R/J \to M$ is also a derivation, and d is the image of $\bar{d}$.

e.g.: $R = \dfrac{\mathbb{C}[x,y]}{(x^3+y^2)}$,

$$\frac{(x^3+y^2)}{(x^3+y^2)^2} \xrightarrow{d} \Omega_{\mathbb{C}[x,y]/\mathbb{C}} \otimes \frac{\mathbb{C}[x,y]}{x^3+y^2} \to \Omega_{R/\mathbb{C}} \to 0$$

So $$\Omega_{R/\mathbb{C}} \cong \frac{\dfrac{\mathbb{C}[x,y]}{(x^3+y^2)}dx \oplus \dfrac{\mathbb{C}[x,y]}{(x^3+y^2)}dy}{3x^2dx + 2y\,dy}$$

Globalize.

Def Let $X \to S$ be a morphism of schemes. Consider the diagonal map $\Delta : X \to X \times_S X$. Recall that the diagonal is a locally closed immersion. Choose an open $U \subseteq X \times_S X$ s.t. we have a factorization $X \xrightarrow[\text{closed}]{\Delta_U} U \to X \times_S X$.

Let $\mathcal{I}_U = \mathrm{Ker}(\mathcal{O}_U \to \Delta_* \mathcal{O}_X)$

Define $\Omega_{X/S} = \Delta^*(\mathcal{I}_U/\mathcal{I}_U^2)$ - This $\mathcal{O}_X$-mod sheaf of Kähler differentials of the S-scheme X.

Rmk. $\Omega_{X/S}$ indeed does not depend on the choice of U.
For $U \underset{\text{open}}{\subseteq} U' \underset{\text{open}}{\subseteq} X \times_S X$ such $\Delta_U : X \to U$, $\Delta_{U'} : X \to U'$ are closed immersions, note $\mathcal{I}_U = \mathcal{I}_{U'}|_U$. So $\mathcal{I}_U/\mathcal{I}_U^2 = \mathcal{I}_{U'}/\mathcal{I}_{U'}^2|_U$
Let $i : U \hookrightarrow U'$ be the inclusion, $\Delta_{U'} = i \circ \Delta_U$

$$(\Delta_{U'})^*(\mathcal{I}_{U'}/\mathcal{I}_{U'}^2) = \Delta_U^* \circ i^*(\mathcal{I}_{U'}/\mathcal{I}_{U'}^2) = \Delta_U^*(\mathcal{I}_U/\mathcal{I}_U^2)$$

Let $i : U \hookrightarrow U'$ be the inclusion, $\Delta_{U'} = i \cdot \Delta_U$

$$(\Delta_{U'})^*(\mathcal{I}_{U'}/\mathcal{I}_{U'}^2) = \Delta_U^* \cdot i^*(\mathcal{I}_{U'}/\mathcal{I}_{U'}^2) = \Delta_U^*(\mathcal{I}_U/\mathcal{I}_U^2)$$

<u>Prop.</u> There is a $f^{-1}(\mathcal{O}_S)$ linear derivation $d_{X/S} : \mathcal{O}_X \longrightarrow \Omega_{X/S}$ such that for any $f^{-1}(\mathcal{O}_S)$ linear derivation d factors uniquely through $d_{X/S}$

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "d"] \arrow[dr, "d_{X/S}"'] & \mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X} \\ & \Omega_{X/S} \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

So $(\Omega_{X/S}, d_{X/S})$ is the unique (upto isom) with this property.

We only mention what $d_{X/S}$ is. The main point is:

<u>Prop.</u> $\Omega_{X/S}$ is a quasi-coherent sheaf.

<u>Pf.</u> Fix an $U \underset{\text{open}}{\subseteq} X \times_S X$ s.t we have factorization

$$X \xrightarrow[\text{closed}]{\Delta_U} U \xrightarrow[\text{open}]{} X \times_S X$$

Then $\mathcal{I}_U$ is q.coh since $\mathcal{I}_U = \mathrm{Ker}(\mathcal{O}_U \to (\Delta_U)_* \mathcal{O}_X)$ and Δ_U is a closed immersion. So $(\Delta_U)_* \mathcal{O}_X$ is q.coh. ◻

<u>Description of $d_{X/S}$:</u> Choose affine open coverings $\{U_i\}_{i \in I}$ of X and $\{V_j\}_{j \in J}$ of S such that each U_i maps to some V_j.

Then the map $X \longrightarrow X \times_S X$ restricted to U_i is the same as $U_i \longrightarrow U_i \times_{V_j} U_i$ where U_i maps to V_j. Thus $\Omega_{X/S}|_{U_i} \xrightarrow{\sim} \widetilde{\Omega}_{\mathcal{O}_X(U_i)/\mathcal{O}_S(V_j)}$

$d_{X/S}|_{U_i}$ is induced by the universal derivation

$$d : \mathcal{O}_X(U_i) \longrightarrow \Omega_{\mathcal{O}_X(U_i)/\mathcal{O}_S(V_j)}$$

<u>Prop.</u> Let $X \to S$ be a finite type morphism, S is noetherian. Then $\Omega_{X/S}$ is a coherent sheaf.

Proof follows from the more general observation and that $\Omega_{X/S}$ is quasi-coherent.

<u>Prop.</u> Let A be a commutative ring. R be a finite type A-alg. Then $\Omega_{R/A}$ is a finitely gen R-mod.

<u>Pf</u> Write $R = A[x_1 \ldots x_n]/I$.

The conormal seq gives a sur

Pf Write $R = A[x_1 \dots x_n]/I$.
The conormal seq gives a surj

$$\Omega_{A[x_1,\dots,x_n]/I/A} \otimes_A R \twoheadrightarrow \Omega_{R/A}$$
$$\xrightarrow{\sim} \bigoplus_{i=1}^{n} R\,dx_i$$

Regularity vs Kähler differentials

Prop Let (R,m) a noetherian local ring containing a field k such that the composition
$$k \to R \to R/m \text{ is an isomorphism}$$
Then the natural map $m/m^2 \to \Omega_{R/k} \otimes_R R/m$ is an isom.

Pf The conormal seq gives an exact seq
$$m/m^2 \longrightarrow \Omega_{R/k} \otimes_R R/m \to \Omega_{R/m/k} \to 0$$
Since $\Omega_{R/m/k} = 0$, we have a surjective map
$$m/m^2 \longrightarrow \Omega_{R/k} \otimes_R R/m$$
We claim that this map is also injective. For that take an $R/m = k$ mod M. We check that the induced map
$$\mathrm{Der}_k(R,M) = \mathrm{Hom}_k(\Omega_{R/k} \otimes_R R/m, M) \to \mathrm{Hom}_k(m/m^2, M)$$
is surj. Here M is thought of as an R-mod by $R \to R/m$.

Fix a k-lin map $\varphi : m/m^2 \to M$.

Note that the inclusion map $k \to R$ gives a k-linear section (a right inverse) of the projection. Thus as a k-vector space $R = k \oplus m$ and $R/m^2 = k \oplus m/m^2$.

Define $d\varphi : R/m^2 \longrightarrow M$ by setting $d\varphi|_k = 0$ and $d\varphi(m/m^2) = \varphi$

Claim: $d\varphi$ is a k-linear derivation.

Pf k-linearity is clear. We check that $d\varphi$ satisfy the Leibniz rule.

For $x, y \in R$, write $\bar{x} = \lambda_x + \alpha_x \in R/m^2$
$$\bar{y} = \lambda_y + \alpha_y \in R/m^2$$
where $\lambda_x, \lambda_y \in k$, $\alpha_x, \alpha_y \in m/m^2$

Then $\bar{x}\cdot\bar{y} = \lambda_x \alpha_y + \lambda_y \alpha_x + \lambda_x \lambda_y$

So $d\varphi(\bar{x}\cdot\bar{y}) = \lambda_x\, d\varphi(\alpha_y) + \lambda_y\, d\varphi(\alpha_x)$ $[\because d\varphi|_k = 0]$
$$= (\lambda_x + \alpha_x)\, d\varphi(\lambda_y + \alpha_y)$$
$$+ (\lambda_y + \alpha_y)\, d\varphi(\lambda_x + \alpha_x) \; [\because d\varphi|_k = 0$$

$= (\lambda x + \alpha_x)\, d\varphi (\lambda y + \alpha y)$
$+ (\lambda y + \alpha_y)\, d\varphi(\lambda x + \alpha_x)$ [$\because d\varphi|_k = 0$, $m \cdot M = 0$]

$= \bar{x}\, d\varphi(\bar{y}) + \bar{y} \cdot d\varphi(\bar{x})$ ☒

Since $d\varphi$ maps to φ, we get the surjectivity.

Prop. Let $L \supseteq k$ be a finite field extension.
$\Omega_{L/k} = 0 \iff L \supseteq k$ is separable. (HW)

End of 04.12.24 lecture

Prop: Let k be an algebraically closed field. $L \supseteq k$ be a finitely generated field extension (i.e. L is the quotient field of some finite type k-alg)
Then $\dim_L \Omega_{L/k} = \operatorname{trdeg}_k L$

Pf Let $d = \operatorname{trdeg}_k L$. Since k is algebraically closed, we can find a transcendence basis L/k, $x_1, \ldots, x_d \in L$ such that

$$k(x_1 \ldots x_d) \subseteq L \text{ (finite) separable.}$$

We have an exact seq

$$\Omega_{k(x_1 \ldots x_d)/k} \otimes L \longrightarrow \Omega_{L/k} \longrightarrow \Omega_{L/k(x_1 \ldots x_d)} \longrightarrow 0$$

Since $\Omega_{L/k(x_1 \ldots x_d)} = 0$, and $\Omega_{k(x_1 \ldots x_d)/k} \otimes_{k(x_1 \ldots x_d)} L \xrightarrow{\sim} L^{\oplus d}$

Choose a f.t k-alg A s.t $\operatorname{Frac}(A) = L$.
Then $\Omega_{L/k} \xrightarrow{\sim} \Omega_{A/k} \otimes_A \operatorname{Frac}(A)$. Since $\Omega_{A/k}$ is f.g $/A$,
$\exists$ $0 \neq f \in A$ such that $\Omega_{A/k} \otimes_A A[1/f] \simeq \Omega_{A_f/k}$ is free.

Take a maximal ideal m of A_f. Note $k \to A_m \to A_m$ is an isom as k is alg closed.
Thus $m/m^2 \simeq (\Omega_{A_f/k})_m \otimes_A A/m$
$\Rightarrow \operatorname{rk}_{A_f} \Omega_{A_f/k} = \dim_{A/m} (\Omega_{A_f/k})_m \otimes A/m = \operatorname{rk}_{A/m} m/m^2$
But $\operatorname{rk}_{A/m} m/m^2 \geq \dim A_m = \dim A = \operatorname{tr}_k L$
So $\operatorname{rk}_L \Omega_{L/k} = \operatorname{rk}_{A_f} \Omega_{A_f/k} = \dim_k L$.

Thm: Let k be an algebraically closed field. X/k be a finite type integral scheme. Then X is regular at x (i.e $\mathcal{O}_{X,x}$ is a regular ring) iff $(\Omega_{X/k})_x$ is a free $\mathcal{O}_{X,x}$ module

Pf. Since the involved statements are local at x, W.L.O.G we can assume $X = \operatorname{Spec}(R)$, where R is a finite type k-alg, which is a domain. and x corresponds to the prime ideal $p \subset R$

W.L.O.G we can assume $X = \mathrm{Spec}(R)$, where R is a finite type k-alg, which is a domain and x corresponds to the prime ideal $p \subseteq R$

Consider the conormal sequence

$$(*) \quad \frac{pR_p}{p^2R_p} \longrightarrow \Omega_{R_p/k} \otimes_{R_p} \frac{R_p}{pR_p} \longrightarrow \Omega_{k(p)/k} \to 0$$ where $k(p) = \frac{R_p}{pR_p} = \mathrm{Frac}(R/p)$.

Assume R_p is regular. Then $\dim_{k(p)} \frac{pR_p}{p^2R_p} = \dim R_p$

By the Lemma above, $\dim_{k(p)} \Omega_{k(p)/k} = \dim R/p$
$= \mathrm{tr.deg}_k (k(p))$

So $\dim_{k(p)} \Omega_{R_p/k} \otimes k(p) \leq \dim R/p + \dim R_p$
$= \dim(X)$ [$\because R$ is f.t./k and R is a domain]

We claim that the above inequality is in fact an equality. Indeed, the L.H.S is the min # of gens of the R_p-mod $\Omega_{R_p/k}$. In case of strict inequality $\Omega_{R_p/k}$ is gen $/R_p$ by $x_1, x_2, \ldots, x_n$ where $n < \dim X$.
But $\Omega_{\mathrm{Frac}(R)/k} = \Omega_{R_p/k} \otimes_{R_p} \mathrm{Frac}(R_p)$ is then gen by the images of $x_1, \ldots, x_n$ in the localization. This contradicts the Prop above.
The freeness of $\Omega_{R_p/k}$ now follows from the lemma below.

__Lemma:__ Let (A, m) be a noetherian domain, M be a finitely generated A-module such that
$$\dim_{A/m} M/mM = \dim_{\mathrm{Frac}(A)} M \otimes_A \mathrm{Frac}(A)$$
Then M is a free module

__Pf.__ Let $\mu = \dim_{A/m} M/mM$.

Consider an exact seq $0 \to K \to A^{\oplus \mu} \to M \to 0$
where the map $A^{\oplus \mu} \to M$ comes from a choice of min set of generators of M.
Applying $\otimes_A \mathrm{Frac}(A)$ to the above exact seq, get $K \otimes_A \mathrm{Frac}(A) = 0$.
But A is a domain $\Rightarrow$ K is torsion free
So $K \otimes_A \mathrm{Frac}(A) = 0 \Rightarrow K = 0$. $\square$

Now assume $\Omega_{R_p/k} \simeq (\Omega_{R/k})_p$ is free.
Then there is $f \notin p$ s.t. $\Omega_{R/k} \otimes_R R[1/f]$ is locally free.

Choose a maximal ideal $m \in R$ of $R[1/f]$ that contains p. $\Omega_{R_m/k}$ is locally free of rk $= \dim X = \dim R_m$
But $\Omega_{R_m/k} \otimes_{R_m} \frac{R_m}{mR_m} \xrightarrow{\sim} m/m^2$
($\because k$, as k is alg closed)

($\overset{\shortparallel}{k}$, as k is alg closed)

So $\dim R_m = \mathrm{rk}_{R_m}(\Omega_{R_m/k}) = \dim m/m^2$

$\Rightarrow R_m$ is regular

Now we use the (non-trivial) fact that localization of a regular local ring regular to conclude $R_p \cong (R_m)_{pR_m}$ is regular.

<u>Prop.</u> Let X be a finite type integral scheme over an algebraically closed field

Then Regular locus of X $:= \{x \in X \mid X$ is reg at x, i.e $\mathcal{O}_{X,x}$ is reg$\}$

$$= \{x \in X \mid (\Omega_{X/k})_x \text{ is free}\}$$

In particular 1) The regular locus of X is open in X

2) X is regular $\Leftrightarrow$ $\Omega_{X/k}$ is a locally free module of $\mathrm{rk} = \dim X$.

<u>H.W</u>: Derive Jacobian criterion from the results above.